13.3: 8, 16, 20, 26, 29–32

8. $\partial(1 + 2xy + \ln x)/\partial y = 2x = \partial(x^2)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x, y) \mid x > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = 1 + 2xy + \ln x$ implies $f(x, y) = x + x^2y + x\ln x - x + g(y)$ and $f_y(x, y) = x^2 + g'(y)$. But $f_y(x, y) = x^2$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$. Then $f(x, y) = x^2y + x\ln x + K$ is a potential function for $\mathbf{F}$.

16. (a) $f_x(x, y, z) = 2xz + y^2$ implies $f(x, y, z) = x^2z + xy^2 + g(y, z)$ and so $f_y(x, y, z) = 2xy + g_y(y, z)$. But $f_y(x, y, z) = 2xy$ so $g_y(y, z) = 0 \quad \Rightarrow \quad g(y, z) = h(z)$. Thus $f(x, y, z) = x^2z + xy^2 + h(z)$ and $f_z(x, y, z) = x^2 + h'(z)$. But $f_z(x, y, z) = x^2 + 3z^2$, so $h'(z) = 3z^2 \quad \Rightarrow \quad h(z) = z^3 + K$. Hence $f(x, y, z) = x^2z + xy^2 + z^3$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0, 1, -1)$ and $t = 1$ corresponds to $(1, 2, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, -1) = 6 - (-1) = 7$.

20. Here $\mathbf{F}(x, y) = (1 - ye^{-x})\,\mathbf{i} + e^{-x}\,\mathbf{j}$. Then $f(x, y) = x + ye^{-x}$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C (1 - ye^{-x})\,dx + e^{-x}\,dy = \int_C \mathbf{F} \cdot d\,\mathbf{r} = f(1, 2) - f(0, 1) = (1 + 2e^{-1}) - 1 = 2/e$.

26. $\nabla f(x, y) = \cos(x - 2y)\,\mathbf{i} - 2\cos(x - 2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0, 0) = \sin 0 = 0$ and $f(\pi, \pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0, 0)$ to (π, π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j}$, $0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0, 0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2}t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0, 0)$ to $\left(\frac{\pi}{2}, 0\right)$.

29. $D = \{(x, y) \mid x > 0, y > 0\}$ = the first quadrant (excluding the axes).

(a) D is open because around every point in D we can put a disk that lies in D.

(b) D is connected because the straight line segment joining any two points in D lies in D.

(c) D is simply-connected because it's connected and has no holes.

30. $D = \{(x, y) \mid x \neq 0\}$ consists of all points in the xy-plane except for those on the y-axis.

(a) D is open.

(b) Points on opposite sides of the y-axis cannot be joined by a path that lies in D, so D is not connected.

(c) D is not simply-connected because it is not connected.

31. $D = \{(x, y) \mid 1 < x^2 + y^2 < 4\}$ = the annular region between the circles with center $(0, 0)$ and radii 1 and 2.

(a) D is open.

(b) D is connected.

(c) D is not simply-connected. For example, $x^2 + y^2 = (1.5)^2$ is simple and closed and lies within D but encloses points that are not in D. (Or we can say, D has a hole, so is not simply-connected.)

32. $D = \{(x, y) \mid x^2 + y^2 \le 1 \text{ or } 4 \le x^2 + y^2 \le 9\}$ = the points on or inside the circle $x^2 + y^2 = 1$, together with the points on or between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(a) D is not open because, for instance, no disk with center $(0, 2)$ lies entirely within D.

(b) D is not connected because, for example, $(0, 0)$ and $(0, 2.5)$ lie in D but cannot be joined by a path that lies entirely in D.

(c) D is not simply-connected because, for example, $x^2 + y^2 = 9$ is a simple closed curve in D but encloses points that are not in D.