13.6: 20, 24, 34, 38; One More Problem

20. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad \mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(u,v)) = u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + v^2\,\mathbf{k}$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^\pi\int_0^1 (u\sin^2 v - u\cos^2 v + uv^2)\,du\,dv = \int_0^\pi\int_0^1(-u\cos 2v + uv^2)\,du\,dv$$
$$= \int_0^\pi \left[-\tfrac{1}{2}\cos 2v + \tfrac{1}{2}v^2\right] dv = \tfrac{1}{6}\pi^3.$$

24. $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$

Using spherical coordinates, S is given by $x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$, $z = 5\cos\phi$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = (5\sin\phi\cos\theta)(5\cos\phi)\,\mathbf{i} + (5\sin\phi\cos\theta)\,\mathbf{j} + (5\sin\phi\sin\theta)\,\mathbf{k}$ and $\mathbf{r}_\phi \times \mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\cos\phi\sin\phi\,\mathbf{k}$, so

$$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta$$

Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)]\,dA$$
$$= \int_0^\pi\int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\theta\,d\phi$$
$$= 125\int_0^\pi \left[5\sin^3\phi\cos\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + \sin^3\phi\left(\tfrac{1}{2}\sin^2\theta\right) + \sin^2\phi\cos\phi\,(-\cos\theta)\right]_{\theta=0}^{\theta=\pi} d\phi$$
$$= 125\int_0^\pi \left(\tfrac{5}{2}\pi\sin^3\phi\cos\phi + 2\sin^2\phi\cos\phi\right) d\phi = 125\left[\tfrac{5}{2}\pi\cdot\tfrac{1}{4}\sin^4\phi + 2\cdot\tfrac{1}{3}\sin^3\phi\right]_0^\pi = 0$$

34. S is given by $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2+y^2}\,\mathbf{k}$, $|\mathbf{r}_x\times\mathbf{r}_y| = \sqrt{1 + \dfrac{x^2+y^2}{x^2+y^2}} = \sqrt{2}$ so

$$m = \iint_S \left(10 - \sqrt{x^2+y^2}\right) dS = \iint_{1\le x^2+y^2\le 16} \left(10 - \sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$
$$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10-r)\,r\,dr\,d\theta = 2\pi\sqrt{2}\left[5r^2 - \tfrac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\,\pi$$

38. A parametric representation for the hemisphere S is $\mathbf{r}(\phi,\theta) = 3\sin\phi\cos\theta\,\mathbf{i} + 3\sin\phi\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0\le\phi\le\pi/2$, $0\le\theta\le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\cos\theta\,\mathbf{i} + 3\cos\phi\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi\times\mathbf{r}_\theta = 9\sin^2\phi\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\cos\phi\,\mathbf{k}$. The rate of flow through S is

$$\iint_S \rho\mathbf{v}\cdot d\mathbf{S} = \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j})\cdot(9\sin^2\phi\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\cos\phi\,\mathbf{k})\,d\theta\,d\phi$$
$$= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\sin\theta\cos\theta + \sin^3\phi\sin\theta\cos\theta\right)d\theta\,d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{2\pi}\sin\theta\cos\theta\,d\theta$$
$$= 54\rho\left[-\tfrac{1}{3}\left(2+\sin^2\phi\right)\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0\ \text{kg/s}$$