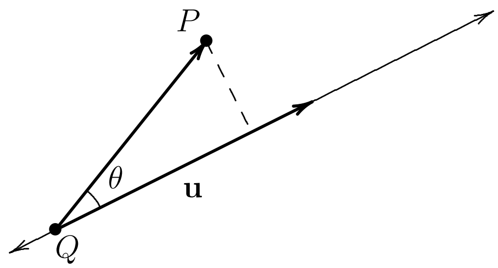
Homework 4: Three More Problems – Solutions

1 (a) Which of the following methods will always give you the distance between two parallel lines L_1 and L_2?

Solution: The answer is

(iii) Pick any point P on line L_1, and find the distance from P to L_2.

Notice that this reduces the problem of finding the distance between two parallel lines to the problem of finding the distance between a point P and a line L_2. If Q is a point on line L_2 and $\mathbf{u}$ is a vector parallel to L_2, then we can think of the distance as the length of one side of a right triangle determined by $\overrightarrow{QP}$ and $\mathbf{u}$ (or the line L_2):



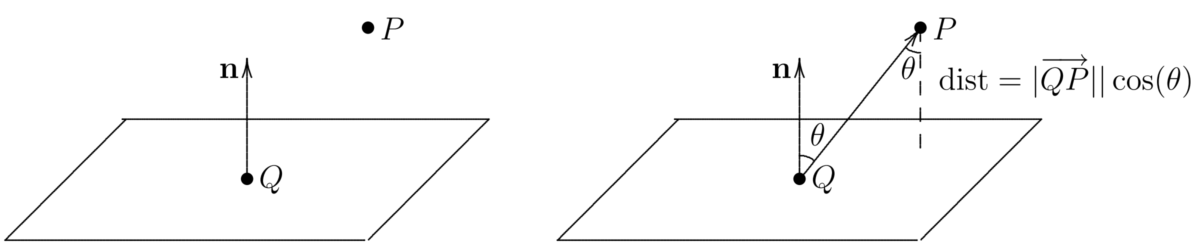
This height is simply $|\overrightarrow{QP}|\sin(\theta)$, which we recognize as most of the length of the cross product of $\overrightarrow{QP}$ with $\overrightarrow{u}$: $|\overrightarrow{QP}|\sin(\theta) = \frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|}$. Therefore, the distance is simply $\frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|}$.

(b) Which of the following methods will always give you the distance between two non-parallel lines L_1 and L_2?

Solution: The answer is

(iv) Pick any point P on line L_1 and any plane $\mathcal{S}$ which contains line L_2 and is parallel to line L_1. Find the distance from P to the plane $\mathcal{S}$.

Notice that this reduces the problem of finding the distance between two non-parallel lines to the problem of finding the distance from a point P to a plane. If Q is a point on the plane and $\mathbf{n}$ is a normal vector for the plane, then the distance from P to the plane is simply the (absolute value of the) scalar projection $|\operatorname{comp}_{\mathbf{n}}\overrightarrow{QP}| = |\overrightarrow{QP}||\cos(\theta)| = \left|\frac{|\mathbf{n}\cdot\overrightarrow{QP}|}{|\mathbf{n}|}\right|$:



(c) Which of the following methods will always give you the distance between two parallel planes $\mathcal{S}_1$ and $\mathcal{S}_2$?

Solution: The answer is

(iv) Pick any point P on plane $\mathcal{S}_1$, and find the distance from P to $\mathcal{S}_2$.

(d) Which of the following methods will always give you the distance between two non-parallel planes $\mathcal{S}_1$ and $\mathcal{S}_2$?

Solution: The answer is

(i) The distance is always 0.

Two non-parallel planes must intersect, so the distance between them is 0.

2 In each part, first decide whether the lines L_1 and L_2 are parallel. (Explain how you know.) Then, use the method you chose in the appropriate part of Problem 1 to find the distance between L_1 and L_2.

(a) L_1 is defined by the parametric vector equation $\mathbf{r}_1(t) = \langle 1,3,5\rangle + t\langle 2,0,1\rangle$, and L_2 is defined by the parametric vector equation $\mathbf{r}_2(t) = \langle 2,7,4\rangle + t\langle 0,-1,1\rangle$.

Solution: L_1 and L_2 are *not parallel*. We know this because L_1 is parallel to $\langle 2,0,1\rangle$ and L_2 is parallel to $\langle 0,-1,1\rangle$, but $\langle 2,0,1\rangle$ and $\langle 0,-1,1\rangle$ are not parallel to each other (they are not scalar multiples of each other).

Therefore, according to Problem 1(b), we should pick a point P on line L_1 and find a plane $\mathcal{S}$ which contains L_2 and is parallel to L_1. Let's use the point $P = (1,3,5)$, which is equal to $\mathbf{r}_1(0)$.

The plane containing L_2 and parallel to L_1 must be parallel to $\langle 2,0,1\rangle$ and $\langle 0,-1,1\rangle$, so a normal vector for the plane is $\mathbf{n} = \langle 2,0,1\rangle \times \langle 0,-1,1\rangle = \langle 1,-2,-2\rangle$. In addition, since the plane should contain line L_2, it must contain the point $Q = (2,7,4)$ (which is equal to $\mathbf{r}_2(0)$).

As explained in the solution to 1(b), above, the distance is then the (absolute value of the) scalar projection $\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP}$. Since the vector $\overrightarrow{QP}$ is equal to $\langle -1,-4,1\rangle$, the scalar projection vector is $\frac{\mathbf{n}\cdot\overrightarrow{QP}}{|\mathbf{n}|} = \frac{5}{3}$. This is thus the distance between L_1 and L_2.

(b) L_1 is defined parametrically by $x = 1+2t$, $y = 3-t$, and $z = 7+3t$. L_2 passes through the point $(5,1,13)$ and is parallel to $\langle 1,3,0\rangle$.

Solution: L_1 passes through the point $(1,3,7)$ and is parallel to $\langle 2,-1,3\rangle$. Since L_2 is parallel to $\langle 1,3,0\rangle$, L_1 and L_2 are *not parallel* (because the vectors $\langle 2,-1,3\rangle$ and $\langle 1,3,0\rangle$ are not parallel). Therefore, we should again use the method described in Problem 1(b).

The point $P = (1,3,7)$ lies on L_1. The plane containing line L_2 and parallel to L_1 is parallel to $\langle 2,-1,3\rangle$ and $\langle 1,3,0\rangle$, so a normal vector for the plane is $\mathbf{n} = \langle 2,-1,3\rangle \times \langle 1,3,0\rangle = \langle -9,3,7\rangle$. A point on the plane is $Q = (5,1,13)$, and $\overrightarrow{QP} = \langle 4,-2,6\rangle$.

The distance between the lines is then (the absolute value of) $\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP} = 0$. So, the distance between the lines is 0. This means that the lines intersect, and we can check that this is true by noticing that $(5,1,13)$ is indeed a point on both lines.

(c) L_1 is defined by the symmetric equations $x - 2 = \frac{y+1}{2} = \frac{3-z}{2}$, and L_2 is defined parametrically by $x = 5-t$, $y = -2t$, and $z = 1+2t$.

Solution: We first rewrite L_1 in parametric form: if $t = x - 2 = \frac{y+1}{2} = \frac{3-z}{2}$, then $x = 2+t$, $y = 2t-1$, and $z = 3-2t$. So, L_1 passes through the point $(2,-1,3)$ and is parallel to the vector $\langle 1,2,-2\rangle$.

L_2 passes through $(5,0,1)$ and is parallel to $\langle -1,-2,2\rangle$. Since the vectors $\langle 1,2,-2\rangle$ and $\langle -1,-2,2\rangle$ are parallel, the two lines are parallel. So, to find the distance between the lines, we use the method described in Problem 1(a). We can use $P = (2,-1,3)$, $Q = (5,0,1)$, and $\mathbf{u} = \langle -1,-2,2\rangle$. Then the vector $\overrightarrow{QP}$ is $\langle -3,-1,2\rangle$, so the distance between L_1 and L_2 is $\frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|} = \sqrt{5}$.

3 In each part, first decide whether the planes $\mathcal{S}_1$ and $\mathcal{S}_2$ are parallel. (Explain how you know.) Then, use the method you chose in the appropriate part of Problem 1 to find the distance between $\mathcal{S}_1$ and $\mathcal{S}_2$.

(a) $\mathcal{S}_1$ has equation $2x+3y+4z=10$, and $\mathcal{S}_2$ has equation $2x+3y-4z=15$.

Solution: From the equations for the planes, we see that $\langle 2,3,4\rangle$ is a normal vector for $\mathcal{S}_1$ and $\langle 2,3,-4\rangle$ is a normal vector for $\mathcal{S}_2$. These two normal vectors are not parallel, so the two planes are *not parallel*.

Therefore, from Problem 1(d), the distance between the two planes is 0 (they intersect).

(b) $\mathcal{S}_1$ has equation $2x+3y+4z=10$, and $\mathcal{S}_2$ has equation $-2x-3y-4z=15$.

Solution: From the equations for the planes, we see that $\langle 2,3,4\rangle$ is a normal vector for $\mathcal{S}_1$ and $\langle -2,-3,-4\rangle$ is a normal vector for $\mathcal{S}_2$. These two vectors are parallel (the second is -1 times the first), so the two planes are *parallel*.

Therefore, from Problem 1(c), we want to pick a point P on $\mathcal{S}_1$ and then find the distance from P to $\mathcal{S}_2$. Let's (arbitrarily) pick the point $P = (5,0,0)$, which is on $\mathcal{S}_1$ because $2\cdot 5 + 3\cdot 0 + 4\cdot 0 = 10$. So, we would now like to find the distance from P to the plane $\mathcal{S}_2$.

We'll use the method described in the solution of Problem 1(b). In this case, $Q = (0,-5,0)$ is a point on $\mathcal{S}_2$, and $\mathbf{n} = \langle -2,-3,-4\rangle$ is a normal vector for $\mathcal{S}_2$. Then $\overrightarrow{QP} = \langle -5,-5,0\rangle$, and the scalar projection is $\operatorname{comp}_{\mathbf{n}}\overrightarrow{QP} = \frac{\mathbf{n}\cdot\overrightarrow{QP}}{|\mathbf{n}|} = \frac{25}{\sqrt{29}}$. This is therefore the distance between the planes $\mathcal{S}_1$ and $\mathcal{S}_2$.