Homework 8: One More Problem – Solutions

1 (a) Suppose we start at the point $(x, y, z) = (0, 0, 1)$ at $t = 0$ and move along the curve until $t = 5$. How far have we traveled along the curve? (Note: this does *not* ask how far the ending point is from the starting point.)

Solution: The answer is $\frac{25\sqrt{5}}{2} \approx 27.9508$ units.
We compute using the arc length formula. Recall that the length of the curve $\mathbf{r}(t)$ on the interval $0 \le t \le 5$ is

$$L = \int_0^5 |\mathbf{r}'(t)| \; dt.$$

Here $\mathbf{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle$ is the vector-valued function from Problem 12 of Section 10.3, so $\mathbf{r}'(t) = \langle 2t, t\sin(t), t\cos(t)\rangle$, which has magnitude $|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2(t) + t^2\cos^2(t)} = \sqrt{5}\; t$. The arc-length is then a very straight-forward integral:

$$L = \int_0^5 \sqrt{5}\, t \; dt = \frac{\sqrt{5}}{2}\; t^2\Big|_0^5 = \frac{25\sqrt{5}}{2}.$$

(b) Suppose we again start at the point $(x, y, z) = (0, 0, 1)$, but now move along the curve in the positive t direction a distance of 3 units. What point have we reached?

Solution: The answer is the point

$$\left(\frac{6}{\sqrt{5}}, \sin\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right) - \frac{\sqrt{6}}{\sqrt[4]{5}}\cos\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right), \cos\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right) + \frac{\sqrt{6}}{\sqrt[4]{5}}\sin\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right)\right) \approx (2.6833, 1.1079, 1.5671)\,,$$

which we reach at $t = \frac{\sqrt{6}}{\sqrt[4]{5}}$.
For this we want to find the interval $0 \le t \le T$ so that the length

$$L = \int_0^T |\mathbf{r}'(t)| \; dt$$

is 3. This is the same integral as in part (a), but it should equal three:

$$3 = \int_0^T \sqrt{5}\, t \; dt = \frac{\sqrt{5}}{2}\; t^2\Big|_0^T = \frac{\sqrt{5}}{2}T^2.$$

Thus $T^2 = \frac{6}{\sqrt{5}}$ and so the time we want is $T = \frac{\sqrt{6}}{\sqrt[4]{5}}$. (Note that we're traveling in the direction of positive t, so our answer is positive.) At this time we have reached the point determined by the position vector

$$\mathbf{r}(T) = \mathbf{r}(\sqrt{6}/\sqrt[4]{5}) = \left\langle \left(\frac{\sqrt{6}}{\sqrt[4]{5}}\right)^2, \sin\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right) - \frac{\sqrt{6}}{\sqrt[4]{5}}\cos\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right), \cos\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right) + \frac{\sqrt{6}}{\sqrt[4]{5}}\sin\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right)\right\rangle$$

$$= \left\langle \frac{6}{\sqrt{5}}, \sin\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right) - \frac{\sqrt{6}}{\sqrt[4]{5}}\cos\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right), \cos\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right) + \frac{\sqrt{6}}{\sqrt[4]{5}}\sin\left(\tfrac{\sqrt{6}}{\sqrt[4]{5}}\right)\right\rangle$$

$$\approx \langle 2.6833, 1.1079, 1.5671\rangle\,.$$

(c) Suppose we once more start at the point $(x, y, z) = (0, 0, 1)$, and again move in the positive t direction. Is there a time $t > 0$ when we have traveled a distance of precisely t units? If so, find this time. If not, explain why not.

Solution: The answer is that this occurs at time $t = \frac{2}{\sqrt{5}} \approx 0.8944$.

This is very similar to part (b). We now want to find the interval $0 \leq t \leq T$ so that the length

$$L = \int_0^T |\mathbf{r}'(t)| \; dt$$

is exactly T, the time we've been traveling. Thus (since the integral is identical to that in (b)) we need $T = \frac{\sqrt{5}}{2}T^2$. We get either $T = 0$ (not surprisingly, but also not what we want) or $T = \frac{2}{\sqrt{5}}$.