

Homework 9: More Problems – Solutions

- 1 This exercise is practice in moving between our three coordinate systems in space: rectangular, cylindrical, and spherical. Write each of the following points in the other two coordinate systems.

(a) $(x, y, z) = (2, -2, 2\sqrt{2})$

Solution: In cylindrical coordinates:

$$(r, \theta, z) = \left(\sqrt{2^2 + (-2)^2}, \tan^{-1}\left(\frac{-2}{2}\right), 2\sqrt{2} \right) = \left(2\sqrt{2}, \frac{7\pi}{4}, 2\sqrt{2} \right).$$

In spherical coordinates:

$$(\rho, \theta, \phi) = \left(\sqrt{2^2 + (-2)^2 + (2\sqrt{2})^2}, \tan^{-1}\left(\frac{-2}{2}\right), \cos^{-1}\left(\frac{z}{\rho}\right) \right) = \left(4, \frac{7\pi}{4}, \frac{\pi}{4} \right).$$

(Care should be taken to find the correct inverse tangent to use for θ . Notice that $(x, y) = (2, -2)$ is in the fourth quadrant, so θ must lie between $\frac{3\pi}{2}$ and 2π .)

(b) $(r, \theta, z) = \left(3, \frac{\pi}{3}, -4\right)$

Solution: In rectangular coordinates:

$$(x, y, z) = \left(3 \cos\left(\frac{\pi}{3}\right), 3 \sin\left(\frac{\pi}{3}\right), -4 \right) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, -4 \right).$$

In spherical coordinates:

$$(\rho, \theta, \phi) = \left(\sqrt{3^2 + (-4)^2}, \frac{\pi}{3}, \cos^{-1}\left(\frac{z}{\rho}\right) \right) = \left(5, \frac{\pi}{3}, \cos^{-1}(-0.8) \right).$$

Note that $\phi = \cos^{-1}(-0.8) \approx 2.4981$ is between $\frac{\pi}{2}$ and π , which is what we expect since $z < 0$.

(c) $(\rho, \theta, \phi) = \left(4, \frac{\pi}{4}, \frac{3\pi}{4}\right)$

Solution: In rectangular coordinates:

$$(x, y, z) = \left(4 \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{\pi}{4}\right), 4 \sin\left(\frac{3\pi}{4}\right) \sin\left(\frac{\pi}{4}\right), 4 \cos\left(\frac{3\pi}{4}\right) \right) = \left(2, 2, -2\sqrt{2} \right).$$

In cylindrical coordinates:

$$(r, \theta, z) = \left(4 \sin\left(\frac{3\pi}{4}\right), \frac{\pi}{4}, 4 \cos\left(\frac{3\pi}{4}\right) \right) = \left(2\sqrt{2}, \frac{\pi}{4}, -2\sqrt{2} \right).$$

(d) $(\rho, \theta, \phi) = (2, \frac{\pi}{3}, \frac{\pi}{4})$

Solution: In rectangular coordinates:

$$(x, y, z) = (2 \sin(\frac{\pi}{4}) \cos(\frac{\pi}{3}), 2 \sin(\frac{\pi}{4}) \sin(\frac{\pi}{3}), 2 \cos(\frac{\pi}{4})) = \left(\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}, \sqrt{2} \right).$$

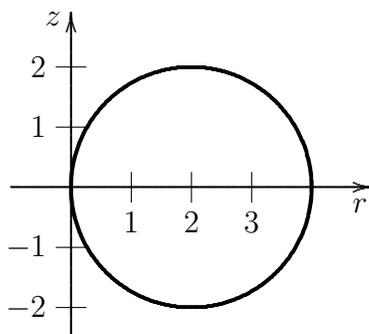
In cylindrical coordinates:

$$(r, \theta, z) = \left(2 \sin(\frac{\pi}{4}), \frac{\pi}{3}, 2 \cos(\frac{\pi}{4}) \right) = \left(\sqrt{2}, \frac{\pi}{3}, \sqrt{2} \right).$$

2 This exercise is more practice in moving between our three coordinate systems in space. Write each of the following surfaces as equations in the other two coordinate systems. Sketch the resulting surface.

(a) $\rho = 4 \sin(\phi)$

Solution: To put this into the other coordinate systems, it's simplest to multiply through by ρ . Then $\rho^2 = 4\rho \sin(\phi)$ is $r^2 + z^2 = 4r$ or $x^2 + y^2 + z^2 = 4\sqrt{x^2 + y^2}$. What is this? The cylindrical coordinate version gives the key: this is a circle $(r - 2)^2 + z^2 = 2^2$ in the rz -plane:



So imagine cutting space with the half-plane $\theta = k$. Then the vertical axis is z and the horizontal axis is r , and so we get this circle. This is true for every possible θ , so we get this picture revolved about the z -axis. Essentially it's a "swollen" donut (swollen because every circle touches the origin, so the hole in the middle is pinched closed).

(b) $r = 2 \cos(\theta), \quad (0 \leq \theta \leq \frac{\pi}{2})$

Solution: Again we use the trick of multiplying by r to get $r^2 = 2r \cos(\theta)$, which in rectangular coordinates is $x^2 + y^2 = 2x$ or (after completing the square) $(x - 1)^2 + y^2 = 1$. Thus this is the cylinder (since z is arbitrary) over the circle in the xy -plane of radius 1 centered at $(1, 0)$.

We can also work in spherical coordinates by writing $r = \rho \sin(\phi)$, so the equation becomes $\rho \sin(\phi) = 2 \cos(\theta)$.

(c) $z = x^2 - y^2$

Solution: This is a hyperbolic paraboloid (the “saddle” quadric). In cylindrical coordinates, this is $z = r^2 (\cos^2(\theta) - \sin^2(\theta))$, or (more simply) $z = r^2 \cos(2\theta)$. We use this to change into spherical coordinates and get $\rho \cos(\phi) = \rho^2 \sin^2(\phi) \cos(2\theta)$ or $\cos(\phi) = \rho \sin^2(\phi) \cos(2\theta)$.

(d) $r = \sqrt{1 + z^2}$

Solution: This is $r^2 = 1 + z^2$ or $x^2 + y^2 - z^2 = 1$, a one-sheeted hyperboloid. To get to spherical coordinates, it might help to write it as $x^2 + y^2 + z^2 = 1 + 2z^2$, so $\rho^2 = 1 + 2\rho^2 \sin^2(\phi)$, or

$$\rho = \frac{1}{\sqrt{1 - 2 \sin^2(\phi)}}.$$

3 This exercise is even more practice in moving between our three coordinate systems in space. Write each of the following solids as inequalities in the other two coordinate systems. Sketch the resulting solid.

(a) $0 \leq \theta \leq \frac{\pi}{2}, \quad r \leq 1$

Solution: This is a solid quarter of a cylinder. In the xy -plane, this is the quarter of the disk $x^2 + y^2 \leq 1$ with $x \geq 0$ and $y \geq 0$. There is no restriction at all involving z , so in space we get everything above and below this quarter disk.

In spherical coordinates this is $0 \leq \theta \leq \frac{\pi}{2}$ and $\rho \sin(\phi) \leq 1$, or $\rho \leq \csc(\phi)$.

(b) $0 \leq \theta \leq \frac{\pi}{2}, \quad \rho \leq 1$

Solution: This is a solid quarter of a solid ball. The solid ball is $x^2 + y^2 + z^2 \leq 1$, and the restrictions on θ imply that $x \geq 0$ and $y \geq 0$.

In cylindrical coordinates, this is $r^2 + z^2 \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$.

(c) $2 \leq \rho \leq 4, \quad \phi \leq \frac{\pi}{3}$

Solution: From $2 \leq \rho \leq 4$, we get a solid ball of radius 4 with a hollow center of radius 2. The restriction on ϕ means we're taking only the part of this hollowed-out ball that is on or above the half-cone $\phi = \frac{\pi}{3}$.

In cylindrical coordinates, this is $2^2 \leq r^2 + z^2 \leq 4^2$ with the further restriction that $z \geq \frac{r}{\sqrt{3}}$ (since $\frac{r}{z} = \tan(\phi) \leq \tan(\frac{\pi}{3}) = \sqrt{3}$).

We can translate this to rectangular coordinates as $2^2 \leq x^2 + y^2 + z^2 \leq 4^2$ with the restriction that $z \geq \sqrt{\frac{x^2 + y^2}{3}}$.

(d) $0 \leq z \leq r$

Solution: The surface $z = r$ is the half-cone $\phi = \frac{\pi}{4}$ (since $\tan(\phi) = \frac{r}{z}$), so this solid is all points between the xy -plane ($z = 0$) and this half-cone. In spherical coordinates this is simply $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$. In rectangular coordinates, it is $0 \leq z \leq \sqrt{x^2 + y^2}$.