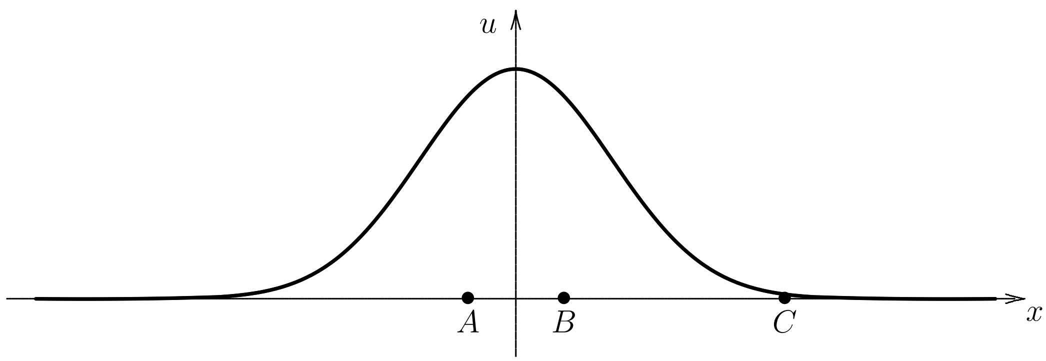
Homework 12: Extra Problems – Solutions

1 The partial differential equation $u_t = au_x$ is known as the *transport equation* (or sometimes the *advection equation*). In this problem we'll find solutions $u(x,t)$ to this equation. In particular, we'll include an initial condition and solve the *initial value problem* (IVP)

$$(*) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} \\ u(x,0) = f(x). \end{cases}$$

(a) Here is a graph of $u = f(x)$ (an example initial condition for the IVP):



This graph is of $u(x,0) = f(x)$. Is the value of $u_x(x,0)$ positive, negative, or zero at the points $x = A$, $x = B$, and $x = C$?

Solution: This is simply asking: for the function $u = f(x)$ graphed above, is $f'(x)$ positive, negative, or zero at the points $x = A$, $x = B$, and $x = C$? From single-variable calculus, we know that $f'(A) > 0$ while $f'(B) < 0$ and $f'(C) < 0$. Thus $u_x(A,0) > 0$ while $u_x(B,0) < 0$ and $u_x(C,0) < 0$.

(b) If $u(x,t)$ is a solution of the IVP labeled $(*)$ for $a > 0$, what is the sign of $u_t(x,0)$ at the points $x = A$, $x = B$, and $x = C$?

Solution: Since $u_t(x,0) = au_x(x,0)$ (and $a > 0$), we know that u_t and u_x are the same sign. Thus $u_t(A,0) > 0$ while $u_t(B,0) < 0$ and $u_t(C,0) < 0$.

(c) One solution to the IVP $(*)$ is $u(x,t) = f(x+at)$. We explore this solution in the next two questions.

(i) Show that $u(x,t) = f(x+at)$ really is a solution to the IVP $(*)$ by calculating $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$.

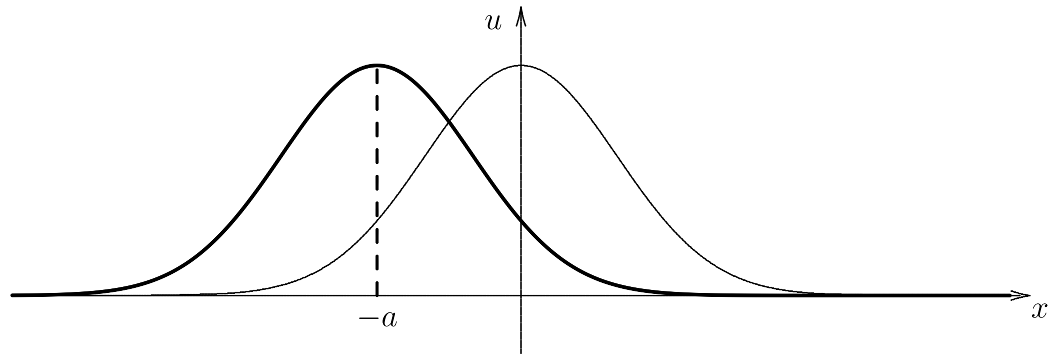
Solution: We compute the two partial derivatives using the chain rule:

$$u_t = \frac{\partial}{\partial t} f(x+at) = f'(x+at)\frac{\partial}{\partial t} f(x+at) = af'(x+at)$$

$$u_x = \frac{\partial}{\partial x} f(x+at) = f'(x+at)\frac{\partial}{\partial x} f(x+at) = f'(x+at).$$

Thus $u_t = au_x$, so $u(x,y) = f(x+at)$ satisfies the given differential equation.

(ii) Sketch the graph of $u(x,t)$ (which, recall, is $f(x+at)$) when $t=1$.
Hint: This is the graph of $f(x+a)$. How does that relate to the graph of $f(x)$ (shown above)?
Solution:



Shown above is the graph of $y=f(x+a)$ with the original $y=f(x)$ sketched lightly in the background. The effect is that $y=f(x)$ has been shifted a units to the left. In particular, the curve $u(x,t)=f(x+at)$ can be thought of as a traveling version of this curve, a version that has been shifted at units to the left after time t.

2 (a) The partial differential equation $u_{tt}=c^2u_{xx}$ is known as the *wave equation*. (You might also call this the one-dimensional wave equation as there is one spatial dimension x with the one temporal dimension t.) Here we'll find some solutions $u(x,t)$ to this equation. In particular, we'll include an initial condition and solve the *initial value problem* (IVP)

$$(**) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} \\ u(x,0)=f(x) \\ u_t(x,0)=0. \end{cases}$$

One solution to the IVP $(**)$ is $u(x,t)=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)$. Again we ask two questions:

(i) Show that $u(x,t)=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)$ really is a solution to the IVP $(**)$.
Solution: The partial derivatives are very similar to those of problem 1:

$$\frac{\partial^2}{\partial x^2}f(x+ct)=f''(x+ct), \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2}f(x-ct)=f''(x-ct),$$

so

$$u_{xx}=\frac{1}{2}\Big(f''(x+ct)+f''(x-ct)\Big);$$

similarly,

$$\frac{\partial^2}{\partial t^2}f(x+ct)=c^2f''(x+ct) \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}f(x-ct)=(-c)^2f''(x-ct)=c^2f''(x-ct),$$

so

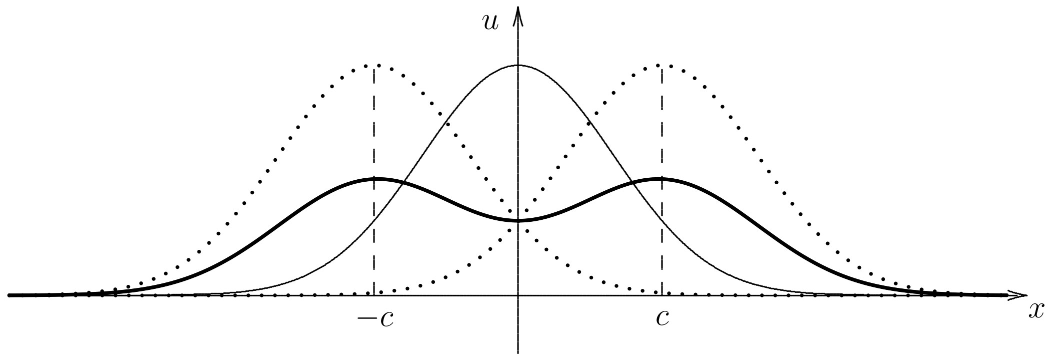
$$u_{tt}=\frac{c^2}{2}\Big(f''(x+ct)+f''(x-ct)\Big).$$

Thus $u_{tt}=c^2u_{xx}$. That this $u(x,t)$ satisfies the given initial conditions is left to you.

(ii) Sketch the graph of $u(x,t)$ for $t = 1$ assuming that $f(x)$ is the same as in the first problem.

Hint: This is just $u(x,1) = \frac{1}{2}\left(f(x+c) + f(x-c)\right)$, the average of two functions. What do the graphs of these two functions look like?

Solution:



The faint line is the original graph $y = f(x)$, and the two dotted graphs are the functions $y = f(x+c)$ and $y = f(x-c)$. The solid dark graph is the average of these two functions (and in particular is bounded between the two dotted curves). In general, this solution will be the average of two graphs, each of which (like the graph in problem 1) is traveling – in this case one to the right and the other to the left. Imagine this as the one-dimensional version of the waves that spread out in a circle, as in a pool of water when something is dropped in the water. In the pool, the wave travels outward in two dimensions (the plane that is the surface of the water); in our case, the wave travels in only one spatial dimension (right and left).

(b) The partial differential equation $u_t = c^2 u_{xx}$ is known as the *heat equation* (or the *diffusion equation* as it covers the diffusion of heat). As before, we'll include an initial condition and consider the *initial value problem* (IVP)

$$(\dagger) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = g(x). \end{cases}$$

(i) Show that one solution to the differential equation is

$$u(x,t) = e^{-c^2\omega^2 t}\big(A\cos(\omega x) + B\sin(\omega x)\big),$$

where c is the constant from the differential equation and A, B, and ω are additional constants.

Solution: The differentiation isn't overly complicated:

$$\begin{aligned} u_t(x,t) &= -c^2\omega^2 e^{-c^2\omega^2 t}\big(A\cos(\omega x) + B\sin(\omega x)\big) \\ u_x(x,t) &= e^{-c^2\omega^2 t}\big(-A\omega\sin(\omega x) + B\omega\cos(\omega x)\big) \\ u_{xx}(x,t) &= e^{-c^2\omega^2 t}\big(-A\omega^2\cos(\omega x) - B\omega^2\sin(\omega x)\big) \\ &= -\omega^2 e^{-c^2\omega^2 t}\big(A\cos(\omega x) + B\sin(\omega x)\big). \end{aligned}$$

Thus $u_t = c^2 u_{xx}$.

(ii) If this $u(x,t)$ is a solution of the IVP labeled (†), what is the function $g(x)$?

Solution:

$$g(x) = u(x,0) = e^{-c^2\omega^2(0)}\big(A\cos(\omega x) + B\sin(\omega x)\big) = A\cos(\omega x) + B\sin(\omega x).$$

(iii) What happens to $u(x,t)$ as t grows without bound? That is, what is $\lim_{t\to\infty} u(x,t)$?

Solution: As t grows without bound, the term $e^{-c^2\omega^2 t}$ decays to zero. Thus the entire function $u(x,t)$ decays as well, so $\lim_{t\to\infty} u(x,t) = 0$.