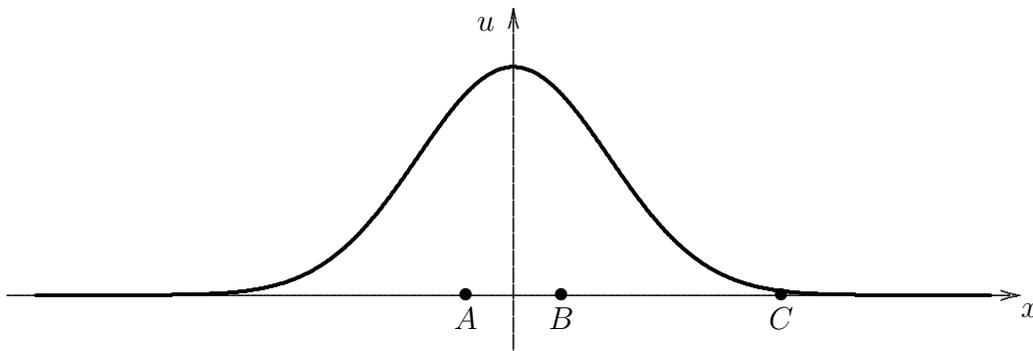


- 1 The partial differential equation $u_t = au_x$ is known as the *transport equation* (or sometimes the *advection equation*). In this problem we'll find solutions $u(x, t)$ to this equation. In particular, we'll include an initial condition and solve the *initial value problem* (IVP)

$$(*) \quad \begin{cases} \frac{\partial u}{\partial t} = a \frac{\partial u}{\partial x} \\ u(x, 0) = f(x). \end{cases}$$

- (a) Here is a graph of $u = f(x)$ (an example initial condition for the IVP):



This graph is of $u(x, 0) = f(x)$. Is the value of $u_x(x, 0)$ positive, negative, or zero at the points $x = A$, $x = B$, and $x = C$?

- (b) If $u(x, t)$ is a solution of the IVP labeled (*) for $a > 0$, what is the sign of $u_t(x, 0)$ at the points $x = A$, $x = B$, and $x = C$?
- (c) One solution to the IVP (*) is $u(x, t) = f(x + at)$. We explore this solution in the next two questions.
- (i) Show that $u(x, t) = f(x + at)$ really is a solution to the IVP (*) by calculating $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$.
- (ii) Sketch the graph of $u(x, t)$ (which, recall, is $f(x + at)$) when $t = 1$.
Hint: This is the graph of $f(x + a)$. How does that relate to the graph of $f(x)$ (shown above)?

- 2 (a) The partial differential equation $u_{tt} = c^2 u_{xx}$ is known as the *wave equation*. (You might also call this the one-dimensional wave equation as there is one spatial dimension x with the one temporal dimension t .) Here we'll find some solutions $u(x, t)$ to this equation. In particular, we'll include an initial condition and solve the *initial value problem* (IVP)

$$(**) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \\ u(x, 0) = f(x) \\ u_t(x, 0) = 0. \end{cases}$$

One solution to the IVP (**) is $u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct))$. Again we ask two questions:

- (i) Show that $u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct))$ really is a solution to the IVP (**).

(ii) Sketch the graph of $u(x, t)$ for $t = 1$ assuming that $f(x)$ is the same as in the first problem.

Hint: This is just $u(x, 1) = \frac{1}{2}(f(x+c) + f(x-c))$, the average of two functions. What do the graphs of these two functions look like?

(b) The partial differential equation $u_t = c^2 u_{xx}$ is known as the *heat equation* (or the *diffusion equation* as it covers the diffusion of heat). As before, we'll include an initial condition and consider the *initial value problem* (IVP)

$$(\dagger) \quad \begin{cases} \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \\ u(x, 0) = g(x). \end{cases}$$

(i) Show that one solution to the differential equation is

$$u(x, t) = e^{-c^2 \omega^2 t} (A \cos(\omega x) + B \sin(\omega x)),$$

where c is the constant from the differential equation and A , B , and ω are additional constants.

(ii) If this $u(x, t)$ is a solution of the IVP labeled (\dagger) , what is the function $g(x)$?

(iii) What happens to $u(x, t)$ as t grows without bound? That is, what is $\lim_{t \rightarrow \infty} u(x, t)$?