Homework 22: A Classic Problem

1 The improper integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges, but it cannot be evaluated using the techniques of single-variable calculus.[1] Surprisingly, this integral *can* be evaluated (pretty easily!) using multi-variable calculus.

Here's how. Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. Of course, we could also call our variable of integration y, so $I = \int_{-\infty}^{\infty} e^{-y^2}\,dy$ as well. If we multiply these two equations, we get

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy.$$

(a) Evaluate the iterated integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$ by converting it to polar coordinates.

(b) Solve for I.

Notice that this problem is essentially a version of Problem 32 in Section 12.4 of the textbook. In particular, the answer is given by Stewart in the formulation of his version of the problem.

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[1] The problem is that we cannot write down an antiderivative of e^{-x^2}. This is not because we're not clever enough but because it can proved that the antiderivative is not an elementary function. Everything that we can write down *is* an elementary function — elementary functions are simply functions made by combining polynomials, trig functions, exponentials, logs, and so on.