1 (5 points) Indicate whether the following statements are True or False by circling the appropriate letter. No justifications are required.

T F The (vector) projection of $\langle 3, 17, -19\rangle$ onto $\langle 1, 2, 3\rangle$ is equal to the (vector) projection of $\langle 3, 17, -19\rangle$ onto $\langle -5, -10, -15\rangle$.

Solution: This is **True**, since $\langle -5, -10, -15\rangle = -5\langle 1, 2, 3\rangle$.

T F The angle between the vectors $\langle 1, -3, 7\rangle$ and $\langle -4, 6, 1\rangle$ is obtuse (greater than $\frac{\pi}{2}$).

Solution: This is **True**. We calculate the cosine of the angle θ between $\mathbf{u} = \langle 1, -3, 7\rangle$ and $\mathbf{v} = \langle -4, 6, 1\rangle$ by

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{(1)(-4) + (-3)(6) + (7)(1)}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{-15}{|\mathbf{u}|\,|\mathbf{v}|} < 0.$$

Since $\cos(\theta) < 0$, the angle θ must be obtuse.

T F If $\mathbf{F} = \langle P, Q\rangle$ is a vector field with the property that $Q_x - P_y = 0$, then Green's Theorem implies that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any curve C.

Solution: This is **False**. Green's Theorem implies that this integral is zero for any *closed* path C, but not for any path at all.

T F The tangent plane to $x^2 - y^2 + 4z^2 = 1$ at the point $(1, 2, 1)$ is $x - 2y + 4z = 1$.

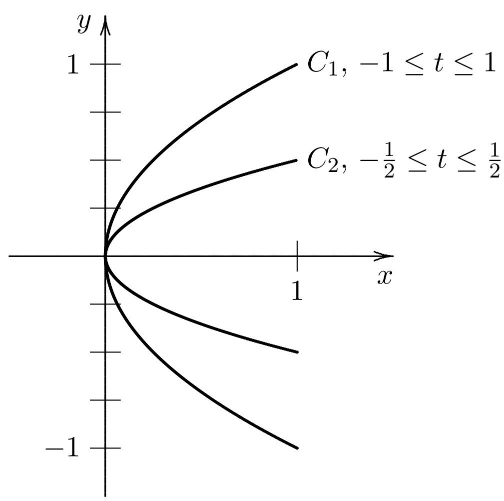
Solution: This is **True**. The normal to the level surface $g(x, y, z) = x^2 - y^2 + 4z^2 = 1$ at the point $(1, 2, 1)$ is $\nabla g(1, 2, 1)$. Since $\nabla g = \langle 2x, -2y, 8z\rangle$, the normal is $\mathbf{n} = \nabla g(1, 2, 1) = \langle 2, -4, 8\rangle$. The tangent plane is therefore

$$\langle 2, -4, 8\rangle\cdot\langle x - 1, y - 2, z - 1\rangle = 0 \quad\text{or}\quad 2x - 4y + 8z = 2 \quad\text{or}\quad x - 2y + 4z = 1.$$

This is the equation given.

T F The two curves C_1, parameterized by $\mathbf{r}_1(t) = \langle t^2, t\rangle$, and C_2, parameterized by $\mathbf{r}_2(t) = \langle 4t^2, t\rangle$, both pass through the point $(0, 0)$. At this point, the curvature of C_1 is greater than the curvature of C_2.

Solution: This is **False**. Here is a graph showing these two parameterized curves:

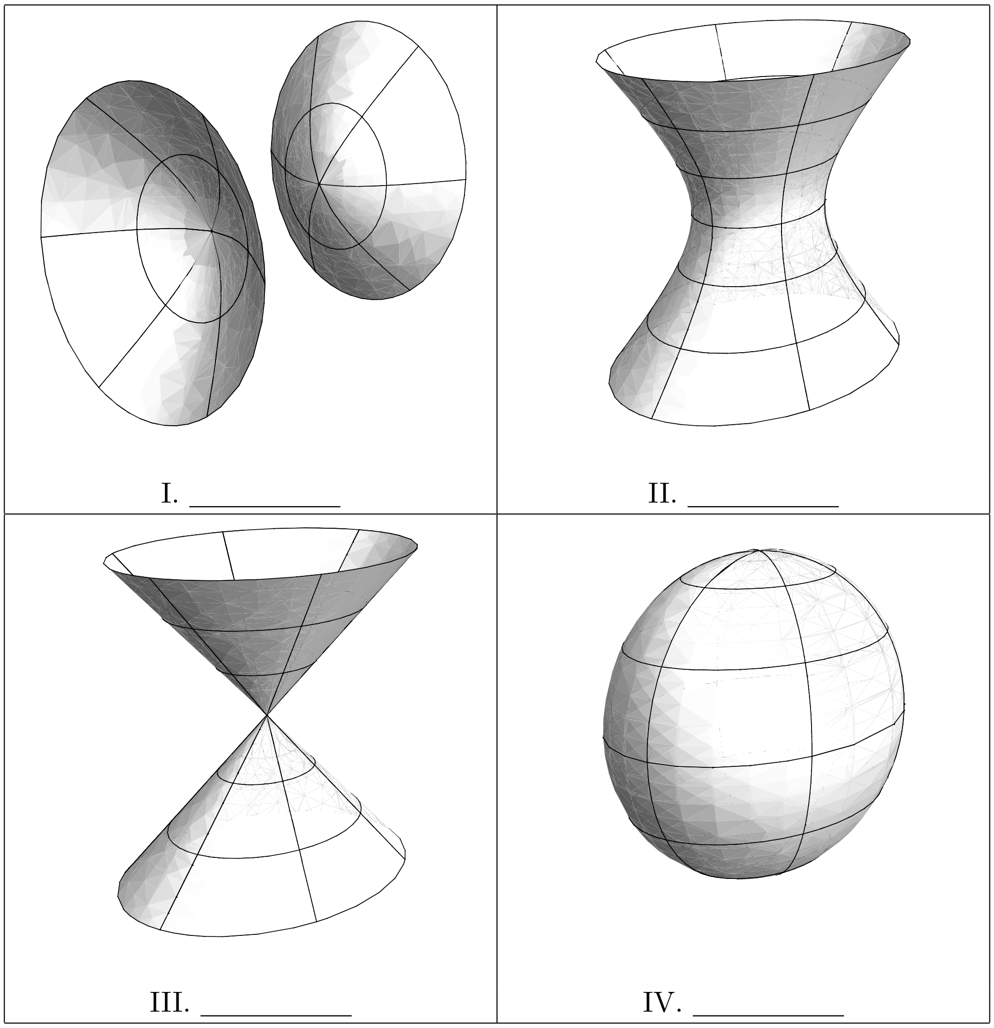


Intuitively, the curve C_2 is curving more at the origin, so has a greater curvature. Here's a rigorous way of thinking of this. The osculating circle for C_1 at the origin is much larger than the same circle for C_2. But the radius for the osculating circle is the inverse of the curvature κ, so this is saying that $\frac{1}{\kappa_1} > \frac{1}{\kappa_2}$. Thus $\kappa_1 < \kappa_2$. Put another way, the curvature for C_1 is *less than* that of C_2, not greater than.

2 (4 points) Match the sketch of each quadric surface with the appropriate equation. You do not need to justify your choices.

(a) $x^2 + 2y^2 + z^2 = 1$ (b) $x^2 + 2y^2 - z^2 = 1$

(c) $x^2 - 2y^2 - z^2 = 1$ (d) $x^2 + 2y^2 - z^2 = 0$



Solution: One way to do this is to look at traces. Here's a little table of traces for our four equations, together with the corresponding sketch:

Equation	$x = k$ trace	$y = k$ trace	$z = k$ trace	Quadric
(a) $x^2 + 2y^2 + z^2 = 1$	ellipses ($\lvert x\rvert \le 1$)	ellipses ($\lvert y\rvert \le \frac{1}{2}$)	ellipses ($\lvert z\rvert \le 1$)	IV
(b) $x^2 + 2y^2 - z^2 = 1$	hyperbolas (all x)	hyperbolas (all y)	ellipses (all z)	II
(c) $x^2 - 2y^2 - z^2 = 1$	ellipses (all $\lvert x\rvert > 1$)	hyperbolas (all y)	hyperbolas (all z)	I
(d) $x^2 + 2y^2 - z^2 = 0$	hyperbolas (all x)	hyperbolas (all y)	ellipses (all $z > 0$)	III

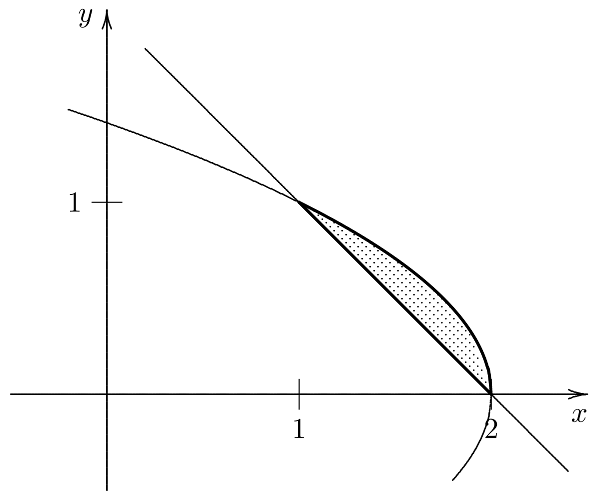
The difference between II and III comes in the trace at $z = 0$ in equation (d). There the ellipse turns into only a point. Another difference is in the $x = 0$ trace. For (b) this is a hyperbola but in (d) it produces two lines $z = \pm\sqrt{2}\, y$ (a "degenerate" hyperbola).

3 (6 points) Compute the integral

$$\int_1^2 \int_{2-x}^{\sqrt{2-x}} \frac{1}{2y^3 - 3y^2 + 4} \, dy \, dx.$$

Hint: Change the order of integration.

Solution: Here's a picture of the region D over which we are integrating. Notice that the y coordinates are bounded below by the line $y = 2 - x$ and above by the hyperbola $y = \sqrt{2-x}$ between $x = 1$ and $x = 2$.



That is, we can write R as

$$R = \{(x, y) \; : \; 2 - x \le y \le \sqrt{2-x}, \; 1 \le x \le 2\}.$$

To change the order of integration, we write x in terms of y for both curves. The line is simply $x = y - 2$ and the parabola is $x = 2 - y^2$. That is, the region R can also be written as

$$R = \{(x, y) \; : \; y - 2 \le x \le 2 - y^2, \; 0 \le y \le 1\}.$$

Thus we can re-write the integral as

$$\int_1^2 \int_{2-x}^{\sqrt{2-x}} \frac{1}{2y^3 - 3y^2 + 4} \, dy \, dx = \int_0^1 \int_{2-y}^{2-y^2} \frac{1}{2y^3 - 3y^2 + 4} \, dx \, dy.$$

Now we integrate:

$$= \int_0^1 \frac{x}{2y^3 - 3y^2 + 4} \bigg|_{2-y}^{2-y^2} dy$$

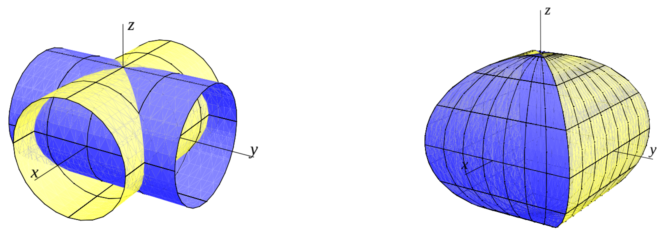
$$= \int_0^1 \frac{(2 - y^2) - (2 - y)}{2y^3 - 3y^2 + 4} \, dy = \int_0^1 \frac{y - y^2}{2y^3 - 3y^2 + 4} \, dy.$$

Now the substitution $u = 2y^3 - 3y^2 + 4$ produces $du = 6(y^2 - y) \, dy$, so we can continue

$$= \int_{y=0}^{y=1} -\frac{1}{6}\frac{1}{u} \, du = -\frac{1}{6} \ln\left(2y^3 - 3y^2 + 4\right)\bigg|_0^1$$

$$= \frac{1}{6}\left(\ln(4) - \ln(3)\right).$$

4 (7 points) Find the volume of the solid enclosed by the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.



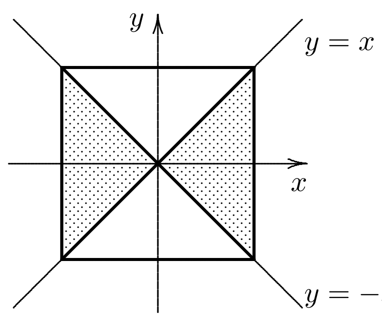
Solution: There are many ways to set up the integral. Let's do it as a triple integral, so if E is the solid, then we want $\iiint_E 1\, dV$. We can describe our solid as $E = \{(x, y, z) : x^2+z^2 \le 1, y^2+z^2 \le 1\}$.

It's hard to write bounds for z in terms of x and y: we know that $z^2 \le 1 - x^2$ and $z^2 \le 1 - y^2$, so we would have to write something like $z^2 \le$ the smaller of $1 - x^2$ and $1 - y^2$, which would be a big mess. We can get around this problem by making the z integral the outer integral. Then we just have to say that z goes from -1 to 1.

Since $x^2 \le 1 - z^2$ and $y^2 \le 1 - z^2$, so x and y both go from $-\sqrt{1-z^2}$ to $\sqrt{1-z^2}$. (That is, the z =constant trace is a square of side length $2\sqrt{1-z^2}$.) This gives an iterated integral

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 1\, dx\, dy\, dz &= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-z^2}\, dy\, dz \\ &= \int_{-1}^{1} 4(1-z^2)\, dz \\ &= 4z - \frac{4z^3}{3}\Big|_{z=-1}^{z=1} \\ &= \frac{16}{3}\end{aligned}$$

Another approach is to split the region up into equally-sized pieces and find the volume of a single, representative piece. We'll cut the region by two planes, $y = x$ and $y = -x$. The advantage here is that the height (the bounds for z) in each region depend on only x (or on only y). Here's the region projected onto the xy-plane:



On the shaded region z varies between $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, while on the unshaded regions z varies between $-\sqrt{1-y^2}$ and $\sqrt{1-y^2}$. We'll integrate over the region R (this is one-quarter of the volume of the full region). Thus the full volume of the region is

$$\text{Volume} = 4\iint_R \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dA = 4\int_0^1\int_{-x}^{x}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dy\, dx = \frac{16}{3}.$$

5 (8 points) Suppose the gradient vector of a function $f(x,y,z)$ at the point $(3,4,-5)$ is $\langle 1,-2,2\rangle$.

(a) (2 points) Find the values of the partial derivatives f_x, f_y, and f_z at the point $(3,4,-5)$.

Solution: Since the gradient vector of a function $f(x,y,z)$ is the vector $\nabla f = \langle f_x, f_y, f_z\rangle$, we have that $f_x = 1$, $f_y = -2$, and $f_z = 2$ at the point $(3,4,-5)$.

(b) (3 points) Find the maximum directional derivative of f at the point $(3,4,-5)$ and the unit vector in the direction in which this maximum occurs.

Solution: Recall that the directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ is greatest in the direction of ∇f. Thus the directional derivative is greatest when $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\langle 1,-2,2\rangle}{3} = \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle$. The directional derivative in the direction of $\mathbf{u}$ is thus $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle 1,-2,2\rangle \cdot \frac{\langle 1,-2,2\rangle}{3} = 3$. (This is, of course, simply $|\nabla f|$.)

(c) (3 points) If $f(3,4,-5) = -2$, estimate $f(3.1, 4.1, -4.8)$ using linear approximation.

Solution: Recall that the linearization $L(x,y,z)$ near $(x,y,z) = (a,b,c)$ of the function $f(x,y,z)$ is

$$L(x,y,z) = f(a,b,c) + \nabla f(a,b,c) \cdot \langle x-a, y-b, z-c\rangle$$

or, equivalently,

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$$

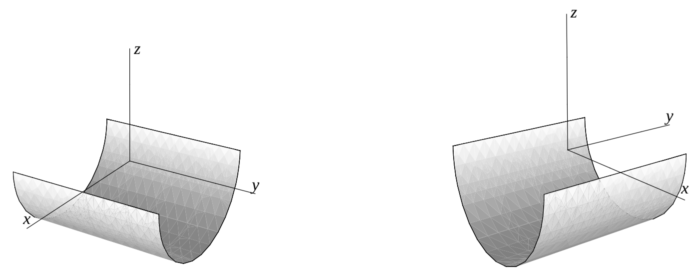
Here $(a,b,c) = (3,4,-5)$, $\nabla f(3,4,-5) = \langle 1,-2,2\rangle$, and $f(3,4,-5) = -2$. Thus

$$L(x,y,z) = -2 + \langle 1,-2,2\rangle \cdot \langle x-3, y-4, z+5\rangle$$

The linear approximation of $f(x,y,z)$ for (x,y,z) near $(3,4,-5)$ is simply $f(x,y,z) \approx L(x,y,z)$. Thus we get

$$\begin{aligned} f(3.1,4.1,-4.8) \approx L(3.1,4.1,-4.8) &= -2 + \langle 1,-2,2\rangle \cdot \langle 3.1-3, 4.1-4, -4.8+5\rangle \\ &= -2 + \langle 1,-2,2\rangle \cdot \langle 0.1, 0.1, 0.2\rangle \\ & -2 + 0.1 - 0.2 + 0.4 = -1.7. \end{aligned}$$

6 (8 points) Let S be the part of the elliptic cylinder $4x^2 + 9z^2 = 36$ lying between the planes $y = -3$ and $y = 3$ and below the plane $z = 0$. Here are two views of this cylinder, from different perspectives:



(a) (4 points) Write an iterated integral which gives the surface area of S. You need not evaluate the integral, but your integral should be simplified enough so that all that is required is integration (that is, the integrand contains no dot or cross products or even vectors).

Solution: Recall that the scalar surface area element dS is $|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$, where $\mathbf{r}(u, v)$ is a parameterization of the surface S. So all we need is a parameterization.

There are several ways to parameterize this. Perhaps the most direct is as a graph $z = f(x, y)$. We solve for z and find that $\mathbf{r}(x, y) = \left\langle x, y, -\frac{1}{3}\sqrt{36 - 4x^2} \right\rangle$ has $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{9+3x^2}{9-x^2}}$, so

$$\text{Surface Area} = \int_{-3}^{3} \int_{-3}^{3} \sqrt{\frac{9 + 3x^2}{9 - x^2}}\, dy\, dx.$$

Note the negative sign in the expression for z and also the general unpleasantness of computing with this expression.

A nicer expression involves parameterizing the elliptical cylinder using sines and cosines: $\mathbf{r}(y, \theta) = \langle 3\cos(\theta), y, 2\sin(\theta)\rangle$ (with $-3 \le y \le 3$ and $\pi \le \theta \le 2\pi$). From this we get

$$\mathbf{r}_y \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -3\sin(\theta) & 0 & 2\cos(\theta) \end{vmatrix} = \langle 2\cos(\theta), 0, 3\sin(\theta)\rangle,$$

and so $|\mathbf{r}_y \times \mathbf{r}_\theta| = \sqrt{9\sin^2(\theta) + 4\cos^2(\theta)}$. Thus

$$\text{Surface Area} = \int_{\pi}^{2\pi} \int_{-3}^{3} \sqrt{9\sin^2(\theta) + 4\cos^2(\theta)}\, dy\, d\theta.$$

(b) (4 points) Let $\mathbf{F}(x, y, z) = \langle x, e^y, z\rangle$. Evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ if S is oriented with normal vectors pointing upward (that is, so the normal vectors have positive z component).

Solution: We use the second parameterization from part (a), namely $\mathbf{r}(y, \theta) = \langle 3\cos(\theta), y, 2\sin(\theta)\rangle$, $\pi \le \theta \le 2\pi$, $-3 \le y \le 3$. From the calculation above, $\mathbf{r}_y \times \mathbf{r}_\theta$ gives us the wrong orientation (recall that $\pi \le \theta \le 2\pi$, so $\sin(\theta) \le 0$), so we use $\mathbf{r}_\theta \times \mathbf{r}_y = \langle -2\cos(\theta), 0, -3\sin(\theta)\rangle$ instead. Thus

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_R \mathbf{F}(\mathbf{r}(y, \theta)) \cdot (\mathbf{r}_\theta \times \mathbf{r}_y)\, dA \\ &= \int_{\pi}^{2\pi} \int_{-3}^{3} \langle 3\cos(\theta), e^y, 2\sin(\theta)\rangle \cdot \langle -2\cos(\theta), 0, -3\sin(\theta)\rangle\, dy\, d\theta \\ &= \int_{\pi}^{2\pi} \int_{-3}^{3} -6\, dy\, d\theta = -36\pi. \end{aligned}$$

7 (14 points) Let C be the curve of intersection of the cylinder $x^2+y^2=4$ and the plane $3x+2y+z=6$.

(a) (2 points) Parameterize C. Be sure to give bounds for your parameter.

Solution: This intersection is an ellipse. It is simplest to parameterize by noticing that its projection into the xy-plane is simply the circle $x^2+y^2=4$. This is parameterized by $x=2\cos(t)$ and $y=2\sin(t)$; the z-coordinate on the ellipse can be found by solving for z in the plane: $z=6-3x-2y$. Thus we end up with the parameterization $\mathbf{r}(t)=\langle 2\cos(t), 2\sin(t), 6-6\cos(t)-4\sin(t)\rangle$ with $0 \le t < 2\pi$.

(b) (3 points) Write an integral that gives the arc length of C. You need not evaluate your integral, but your integral should be simplified enough so that all that is required is integration (that is, the integrand contains no dot or cross products or even vectors).

Solution: Recall that the incremental arc length element is $ds=|\mathbf{r}'(t)|\,dt$. (If we envision $\mathbf{r}(t)$ as describing the path of a particle, then $|\mathbf{r}'(t)|$ is the speed of this particle while $\frac{ds}{dt}$ is the rate of change of the distance this particle has traveled. These must agree.) This is simply

$$\begin{aligned}\text{Arc Length} &= \int_0^{2\pi} \left|\left\langle -2\sin(t), 2\cos(t), 6\sin(t)-4\cos(t)\right\rangle\right|\,dt \\ &= \int_0^{2\pi} \sqrt{4\sin^2(t)+4\cos^2(t)+36\sin^2(t)-24\sin(t)\cos(t)+16\cos^2(t)}\,dt \\ &= \int_0^{2\pi} \sqrt{40\sin^2(t)-24\sin(t)\cos(t)+20\cos^2(t)}\,dt \\ &= \int_0^{2\pi} 2\sqrt{10\sin^2(t)-6\sin(t)\cos(t)+5\cos^2(t)}\,dt.\end{aligned}$$

(c) (4 points) A bee is flying along the curve C in a room where the temperature is given by $f(x,y,z)=5x+5y+2z$. What is the hottest point the bee will reach?

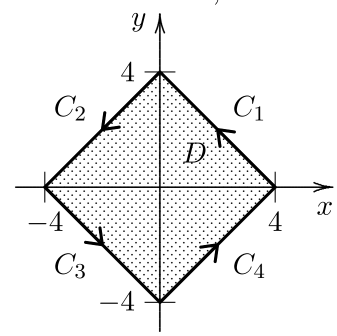
Solution: Maximize $f(\mathbf{r}(t))=10\cos t+10\sin t+12-12\cos t-8\sin t=12-2\cos t+2\sin t$. The derivative of this is $2\sin t+2\cos t$, which is 0 when $\sin t=-\cos t=\pm\frac{1}{\sqrt{2}}$. So, the hottest point is $\left(-\sqrt{2},\sqrt{2},6+\sqrt{2}\right)$ where $f(-\sqrt{2},\sqrt{2},6+\sqrt{2})=12+2\sqrt{2}$. (The other point, $\left(\sqrt{2},-\sqrt{2},6-\sqrt{2}\right)$ where $f(\sqrt{2},-\sqrt{2},6-\sqrt{2})=12-2\sqrt{2}$.

Alternatively, use Lagrange multipliers to maximize $5x+5y+2z$ subject to the constraints $x^2+y^2=4$ and $3x+2y+z=6$. This gives $\langle 5,5,2\rangle=\lambda\langle 2x,2y,0\rangle+\mu\langle 3,2,1\rangle$, so $\mu=2$, and $\langle -1,1,0\rangle=\lambda\langle 2x,2y,0\rangle$. Therefore, $y=-x=\pm\sqrt{2}$, as before.

(d) (5 points) Let $\mathbf{F}(x,y,z)=\left\langle x^2y, e^{y^3}-xy^2, \cos z^2\right\rangle$. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ if C is oriented counterclockwise when viewed from above.

Solution: Use Stokes' Theorem: let S be the part of the plane $3x+2y+z=6$ inside the cylinder $x^2+y^2=4$, oriented with upward normals. Parameterize this by $\mathbf{r}(x,y)=\langle x,y,6-3x-2y\rangle$, where the points (x,y) lie in R, the disk $x^2+y^2\le 4$. Then $\mathbf{r}_x\times\mathbf{r}_y=\langle 3,2,1\rangle$ (which is the correct orientation for Stokes'), and $\operatorname{curl}\mathbf{F}=\langle 0,0,-x^2-y^2\rangle$. Hence

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_R \operatorname{curl}\mathbf{F}(\mathbf{r}(x,y))\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dA \\ &= \iint_R \langle 0,0,-x^2-y^2\rangle\cdot\langle 3,2,1\rangle\,dA = \iint_R -(x^2+y^2)\,dA \\ &= \int_0^{2\pi}\int_0^2 -r^2\cdot r\,dr\,d\theta = -8\pi\end{aligned}$$



8 (6 points) Evaluate the line integral

$$\int_C (x^4 + 3y)\ dx + (5x - y^3)\ dy,$$

where C is the boundary of the square with vertices $(4, 0), (0, 4), (-4, 0)$, and $(0, -4)$, traversed counterclockwise.

Solution: The simplest way to do this is to use Green's Theorem, which says that $\int_C P\ dx + Q\ dy = \iint_D (Q_x - P_y)\ dA$, where C is the oriented boundary of the region D. In our case, this means D is the solid square (as shaded, above), $P = x^4 + 3y$, $Q = 5x - y^3$, and so $Q_x - P_y = 2$. Thus

$$\int_C (x^4 + 3y)\ dx + (5x - y^3)\ dy = \iint_D 2\ dA = 2\ \text{Area}(D).$$

Since D is a square with side length $4\sqrt{2}$, the area of D is 32 and so the line integral is 64.

Another approach is to just compute the line integral directly. I've added labels to the picture (above), so the four line segments are now labeled as C_1 through C_4. We'll do the line integrals along each curve more or less without comment, starting with a parameterization and proceeding to the line integral.

Along C_1: Use $\langle x, y\rangle = \langle 4 - t, t\rangle$ (with $0 \le t \le 4$), so $\langle dx, dy\rangle = \langle -dt, dt\rangle$ and

$$\int_{C_1} (x^4 + 3y)\ dx + (5x - y^3)\ dy = \int_0^4 \left[(4-t)^4 + 3t\right](-dt) + \left[5(4-t) - t^3\right]\ dt = -\frac{1264}{5} = -252.8.$$

Along C_2: Use $\langle x, y\rangle = \langle -t, 4 - t\rangle$ (with $0 \le t \le 4$), so $\langle dx, dy\rangle = \langle -dt, -dt\rangle$ and

$$\int_{C_2} (x^4+3y)\ dx+(5x-y^3)\ dy = \int_0^4 \left[(-t)^4 + 3(4-t)\right](-dt) + \left[5(-t) - (4-t)^3\right]\ (-dt) = -\frac{624}{5} = -124.8.$$

Along C_3: Use $\langle x, y\rangle = \langle t - 4, -t\rangle$ (with $0 \le t \le 4$), so $\langle dx, dy\rangle = \langle dt, -dt\rangle$ and

$$\int_{C_3} (x^4+3y)\ dx+(5x-y^3)\ dy = \int_0^4 \left[(t-4)^4 + 3(-t)\right]\ dt + \left[5(t-4) - (-t)^3\right]\ (-dt) = \frac{784}{5} = 156.8.$$

Along C_4: Use $\langle x, y\rangle = \langle t, t - 4\rangle$ (with $0 \le t \le 4$), so $\langle dx, dy\rangle = \langle dt, dt\rangle$ and

$$\int_{C_4} (x^4 + 3y)\ dx + (5x - y^3)\ dy = \int_0^4 \left[t^4 + 3(t-4)\right]\ dt + \left[5t - (t-4)^3\right]\ dt = \frac{1424}{5} = 284.8.$$

The total line integral around C is the sum of these four line integrals. That is,

$$\int_C (x^4 + 3y)\ dx + (5x - y^3)\ dy = -\frac{1264}{5} - \frac{624}{5} + \frac{784}{5} + \frac{1424}{}5 = \frac{320}{5} = 64,$$

as before.

9 (11 points)

(a) (5 points) Find all critical points of $f(x,y) = x^2+y^2+x^2y+4$, and classify each critical point as a local minimum, local maximum, or saddle point.

Solution: A critical point of $f(x,y)$ is a point where $\nabla f = \langle f_x, f_y\rangle = \langle 0,0\rangle$. Here $\nabla f = \langle 2x+2xy, 2y+x^2\rangle$, so we have $2x(y+1) = 0$ and $2y+x^2 = 0$. From the first equation, either $x = 0$ (and so $y = 0$ from the second equation) or $y = -1$ (which means $x = \pm\sqrt{2}$). Thus we have three critical points: $(x,y) = (0,0)$ and $(\pm\sqrt{2}, -1)$.

To classify these critical points, we use the second derivative test. Using $f_{xx} = 2y+2$, $f_{xy} = f_{yx} = 2x$, and $f_{yy} = 2$, we get $D = f_{xx}f_{yy} - f_{xy}^2 = 4(y+1) - 4x^2$. Thus we have the following classifications:

Critical Point	$(-\sqrt{2}, -1)$	$(0,0)$	$(\sqrt{2}, -1)$
D	-8	4	-8
f_{xx}	0	2	0
Classification	Saddle	Local Min	Saddle

(b) (3 points) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C be the straight line path from $(1,1)$ to $(2,2)$ and $\mathbf{F}(x,y)$ is the gradient vector field of $f(x,y)$ (from part (a)).

Solution: We could parameterize the curve C and compute the integral directly (and we do so below). It is simpler, however, to use the fundamental theorem of line integrals:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(\text{ end pt }) - f(\text{ start pt }) = f(2,2) - f(1,1) = 20 - 7 = 13.$$

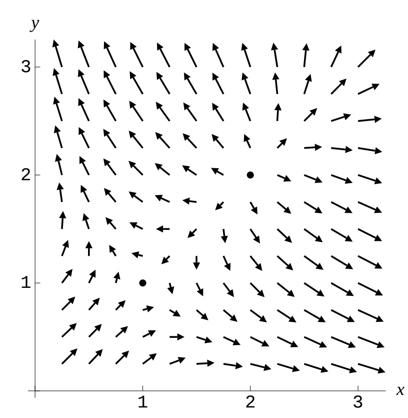
The straightforward-seeming approach is much more complicated. We cab parameterize this path by $\mathbf{r}(t) = \langle t, t\rangle$ (with $1 \le t \le 2$). Using this parameterization, our line integral is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_1^2 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\ &= \int_1^2 \left\langle 2t+2t^2, 2t+t^2\right\rangle \cdot \langle 1,1\rangle\,dt \\ &= \int_1^2 (4t+3t^2)\,dt = 13.\end{aligned}$$

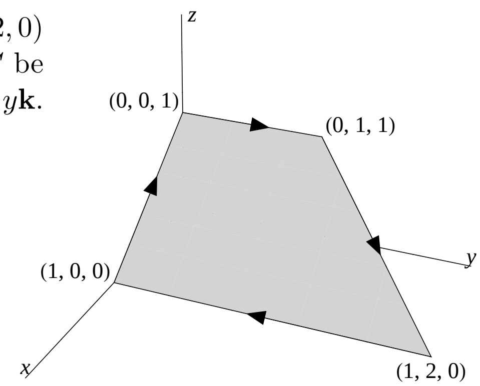
(c) (3 points) Here is the gradient vector field $\mathbf{G}(x,y)$ of another function $g(x,y)$. (Dots represent zero vectors.) Find all critical points of $g(x,y)$ in the region shown, and classify each critical point as a local minimum, local maximum, or saddle point.

Solution: Recall that a critical point of $f(x,y)$ is where $\nabla f = \langle f_x, f_y\rangle = \langle 0,0\rangle$. Thus a critical point is where the vector appears as a dot, so in this region the critical points are at $(x,y) = (1,1)$ and $(2,2)$.

We classify these points using the meaning of the gradient. Recall that the gradient points in the direction in which f increases fastest. The point $(2,2)$ is a minimum because f increases when we move away this point in any direction. The point $(1,1)$ is a saddle point because f either increases or decreases as we move away from the point, depending on the direction.



10 (10 points) The four points $(1,0,0)$, $(0,0,1)$, $(0,1,1)$, and $(1,2,0)$ are all co-planar and form the vertices of a quadrilateral S. Let C be the boundary of S directed as shown to the right, and let $\mathbf{F} = y\mathbf{k}$.



(a) (4 points) Find the equation of the plane that contains the quadrilateral S.

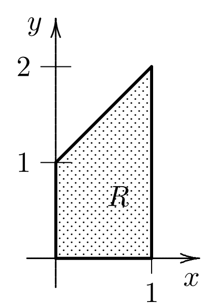
Solution: The plane contains the point $(0,0,1)$ and is parallel to the vectors $\langle 1-0, 0-0, 0-1\rangle$ and $\langle 0-0, 1-0, 1-1\rangle$. Thus the normal is

$$\mathbf{n} = \langle 1,0,-1\rangle \times \langle 0,1,0\rangle = \langle 1,0,1\rangle,$$

and the plane is $\langle 1,0,1\rangle \cdot \langle x-1, y, z\rangle$ or $x+z=1$.

(b) (3 points) Calculate $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Solution: While we could parameterize the four segments of curve C, it is simpler to apply Stokes' Theorem and compute a flux integral instead. We orient the surface S to be compatible with the orientation of C (that is, with the downward pointing normals). A simple parameterization is $\mathbf{r}(x,y) = \langle x, y, 1-x\rangle$ (using the equation of the plane $z = 1-x$ found in part (a)). Then S is the surface parameterized by $\mathbf{r}$ for points (x,y) in the region R, shown below:



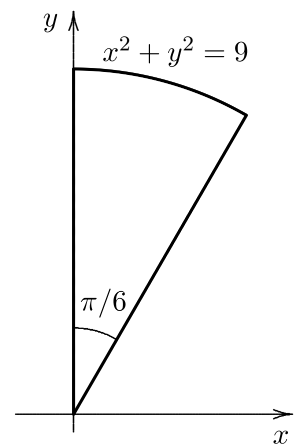
Then $\mathbf{r}_x \times \mathbf{r}_y = \langle 1,0,1\rangle$ is precisely the cross product found in part (a). (Notice this is the wrong orientation; we'll use $\mathbf{r}_y \times \mathbf{r}_x = \langle -1,0,-1\rangle$ instead.) Using $\operatorname{curl}\mathbf{F} = \langle 1,0,0\rangle$, we get via Stokes' Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_R \langle 1,0,0\rangle \cdot \langle -1,0,-1\rangle \, dy\, dx = \int_0^1 \int_0^{1+x} -1 \, dy\, dx = -\frac{3}{2}.$$

(c) (3 points) Calculate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$.

Solution: We've already computed this integral, assuming that S has the orientation we gave it in part (b). If S is given the opposite orientation (with upward-pointing normals), this would simply change the sign of our computation in part (b) and we'd get $+\frac{3}{2}$.

11 (10 points)



(a) (4 points) Write $\iint_D f(x,y)\,dA$, where D is the region shown to the right, as an iterated integral of the form $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

Solution: The line has equation $\frac{y}{x} = \tan(\frac{\pi}{3}) = \sqrt{3}$ or $y = \sqrt{3}x$; this intersects the circle at the point $(x,y) = (3\cos(\frac{\pi}{3}), 3\sin(\frac{\pi}{3})) = (\frac{3}{2}, \frac{3\sqrt{3}}{2})$. Thus the integral can be written as

$$\iint_D f(x,y)\,dA = \int_0^{3/2} \int_{\sqrt{3}x}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx.$$

(b) (4 points) Write $\iint_D f(x,y)\,dA$ as an iterated integral using polar coordinates.

Solution: The region D can be described in polar coordinates as those points with $0 \le r \le 3$ and $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$. Thus

$$\iint_D f(x,y)\,dA = \int_{\pi/3}^{\pi/2} \int_0^3 f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta.$$

(c) (2 points) Compute $\iint_D \cos(x^2+y^2)\,dA$.

Solution: We use polar coordinates as in part (b). We get

$$\begin{aligned}\iint_D \cos(x^2+y^2)\,dA &= \int_{\pi/3}^{\pi/2} \int_0^3 \cos(r^2)\,r\,dr\,d\theta \\ &= \int_{\pi/3}^{\pi/2} \left.\frac{\sin(r^2)}{2}\right|_0^3 d\theta = \int_{\pi/3}^{\pi/2} \frac{\sin(9)}{2}\,d\theta \\ &= \frac{\sin(9)}{2}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{12}\sin(9).\end{aligned}$$

12 (11 points) Let W be the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This tetrahedron has volume $\frac{1}{6}$.

(a) (4 points) Let S be the surface that forms the boundary of W, oriented with the outward-pointing normal. Note that S has four pieces. Compute the flux of the vector field $\mathbf{F}(x,y,z) = (x - 3yz)\mathbf{i} + (x^3 + y)\mathbf{j} + (\tan(xy) + z)\mathbf{k}$ out of the surface S. That is, compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: We use the Divergence Theorem. There are two hints at why this might be appropriate. First, computing the flux directly would require that we parameterize each of the four pieces of S separately. Second, and perhaps even more importantly, the vector field $\mathbf{F}$ is complicated and difficult to involve in an integral, while $\operatorname{div}\mathbf{F} = 3$. The Divergence Theorem implies that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div}\mathbf{F}\, dV = \iiint_W 3\, dV = 3\,\mathrm{Vol}(W) = \frac{1}{2},$$

as we're told the volume of W is $\frac{1}{6}$.

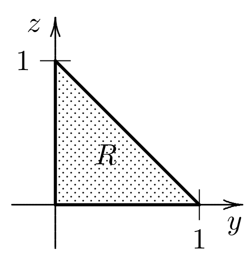
(b) (3 points) Compute the flux of $\operatorname{curl}\mathbf{F}$ out of the surface S; that is, compute $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.

Solution: Again we apply the Divergence Theorem. This time it's even simpler, as we recall that $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for any vector field $\mathbf{F}$:

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div}(\operatorname{curl}\mathbf{F})\; dV = \iiint_W 0\, dV = 0.$$

(c) (4 points) Let S_1 be the part of S that lies in the yz-plane (where $x = 0$), oriented in the same way as S. Find the flux of $\mathbf{F}$ through S_1; that is, compute $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

Solution: We parameterize S_1 by $\mathbf{r}(y,z) = \langle 0, y, z \rangle$ where (y,z) lies in the region R shown below:



The normal $\mathbf{r}_z \times \mathbf{r}_y = \langle -1, 0, 0 \rangle$ is the proper orientation (outward from W), so

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_R \left\langle 0 - 3yz, 0^3 + y, \tan(0y) + z \right\rangle \cdot \langle -1, 0, 0 \rangle\, dz\, dy \\ &= \int_0^1 \int_0^{1-y} 3yz\, dz\, dy = \int_0^1 \tfrac{3}{2} y(y-1)^2\, dy \\ &= \frac{3}{2} \int_0^1 \left(y^3 - 2y^2 + y\right)\, dy = \frac{1}{8}.\end{aligned}$$