

PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

- 1 True-False questions. Circle the correct letter. No justifications are required.
- T F** The divergence of the gradient of any $f(x, y, z)$ is always zero.
- T F** If a nonempty quadric surface $g(x, y, z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.
- T F** If \mathbf{F} is a vector field in space, then the flux of \mathbf{F} through any closed surface S is 0.
- T F** The flux of the vector field $\mathbf{F}(x, y, z) = \langle y + z, y, -z \rangle$ through the boundary of a solid region E is equal to the volume of E .
- T F** If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.
- 2 Which of the following is the cosine of the angle between the two planes $-5x - 3y - 4z = 0$ and $x + y + z = 1$?
- (a) $\frac{6\sqrt{2}}{5}$ (b) $\frac{2\sqrt{5}}{6}$ (c) $\frac{2\sqrt{6}}{5}$
- (d) $\frac{5\sqrt{3}}{2}$ (e) None of the above.
- 3 Which of the functions $u(x, t)$ below are solutions to the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$?
- (a) $u(x, t) = e^{-t} \sin x$ (b) $u(x, t) = e^{-2t} \sin 2x$ (c) $u(x, t) = e^t \sin x$
- (d) $u(x, t) = 3$ (e) $u(x, t) = x^2 + 2t$
- 4 Which of the following line integrals below are path independent?
- (a) $\int_C (10x - 7y) dx - (7x - 2y) dy$
- (b) $\int_C (45x^4y^2 - 6y^6 + 3) dx + (18x^5y - 12xy^5 + 7) dy$
- (c) $\int_C (2e^y - ye^x) dx + (2xe^y - e^x) dy$
- (d) $\int_C 4y^2 \cos(xy^2) dx + 8x \cos(xy^2) dy$
- (e) $\int_C (\sin y + y \sin x) dx + (x \cos y - \cos x + 1) dy$
- 5 The distance between the point $(5, 9, 3)$ and the plane $2x + 4y - z = 1$ is
- (a) $21\sqrt{2}$ (b) $\frac{2}{\sqrt{21}}$
- (c) $\frac{\sqrt{21}}{2}$ (d) $2\sqrt{21}$
- (e) None of the above

6 Consider the following geometric objects,

- $C_1 =$ curve parameterized by $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$, $0 \leq t \leq \pi$
- $C_2 =$ curve parameterized by $\mathbf{r}(t) = \langle \cos(t), -\sin(t), 0 \rangle$, $0 \leq t \leq \pi$
- $S_1 = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$ oriented *upward* at $(0, 0, 1)$
- $S_2 = \{x^2 + y^2 + z^2 = 1, z \leq 0\}$ oriented *upward* at $(0, 0, -1)$
- $S_3 = \{x^2 + y^2 + z^2 = 1\}$ oriented outward, $E = \{x^2 + y^2 + z^2 \leq 1\}$

Which of the statements below are true?

- (a) $\int_{C_1} \langle -y, x, z \rangle \cdot d\mathbf{r} + \int_{C_2} \langle -y, x, z \rangle \cdot d\mathbf{r} = \iint_{S_1} \langle 0, 0, 2 \rangle \cdot d\mathbf{S}.$
- (b) $\iint_{S_1} e^{x^2+2y^2+3z^2} dS + \iint_{S_2} e^{x^2+2y^2+3z^2} dS = \iint_{S_3} e^{x^2+2y^2+3z^2} dS.$
- (c) $\iint_{S_1} \langle y^2, z^2, x^2 \rangle \cdot d\mathbf{S} - \iint_{S_2} \langle y^2, z^2, x^2 \rangle \cdot d\mathbf{S} = 0.$
- (d) $\iint_{S_1} \langle x, -y, z^3 \rangle \cdot d\mathbf{S} = \iint_{S_2} \langle x, -y, z^3 \rangle \cdot d\mathbf{S} + \iiint_E 3z^2 dV.$
- (e) $\iint_{S_1} \langle x^5, z, y \rangle \cdot d\mathbf{S} + \iint_{S_2} \langle x^5, z, y \rangle \cdot d\mathbf{S} = \iiint_E 5x^4 dV.$
- (f) $\int_{C_1} \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle \cdot d\mathbf{r} = \int_{C_2} \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle \cdot d\mathbf{r}$

7 Consider the following tetrahedra,

- (a) The tetrahedron E_1 with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.
- (b) The tetrahedron E_2 with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 3)$.
- (c) The tetrahedron E_3 with vertices $(0, 0, 0)$, $(3, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 1)$.
- (d) The tetrahedron E_4 with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$.

Match each tetrahedron with the integral below that computes

$$\iiint_{E_i} f(x, y, z) dV.$$

- (i) $\int_0^2 \int_0^{(2-x)/2} \int_0^{(6-3x-6y)/2} f(x, y, z) dz dy dx$
- (ii) $\int_0^1 \int_0^{3-3x} \int_0^{(6-6x-2y)/3} f(x, y, z) dz dy dx$
- (iii) $\int_0^1 \int_0^{2-2x} \int_0^{(6-6x-3y)/2} f(x, y, z) dz dy dx$
- (iv) $\int_0^3 \int_0^{(6-2x)/3} \int_0^{(6-2x-3y)/6} f(x, y, z) dz dy dx$

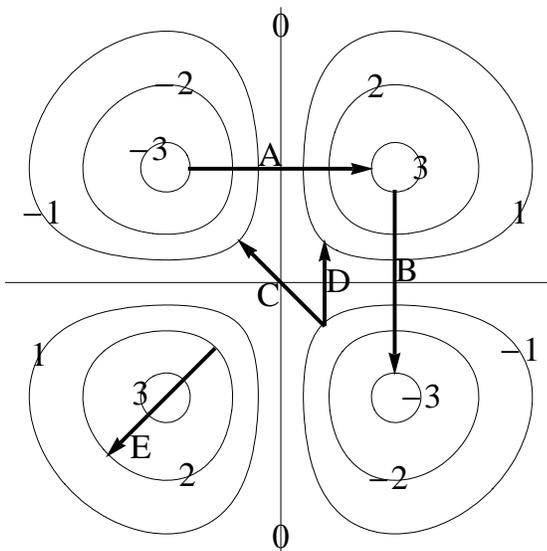
- 8 Consider the surface parameterized by

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle,$$

where $0 \leq u \leq 2$ and $0 \leq v \leq \pi$. Which of the following is an equation of a tangent plane to this surface at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$?

- (a) $x - y - \frac{1}{\sqrt{2}}z = -\frac{1}{\sqrt{2}}$ (b) $-x + y - \frac{1}{\sqrt{2}}z = -\frac{1}{\sqrt{2}}$
 (c) $x + y - \frac{1}{\sqrt{2}}z = \frac{1}{\sqrt{2}}$ (d) $-x - y - \frac{1}{\sqrt{2}}z = -\frac{3}{\sqrt{2}}$
 (e) None of the above.

- 9 Let $f(x, y)$ be a function in the plane having level sets as shown in the plot below and assume that f has continuous derivatives defined everywhere.



Give the value of each of the following integrals.

- (a) $\int_A \nabla f \cdot d\mathbf{r}$ (b) $\int_B \nabla f \cdot d\mathbf{r}$ (c) $\int_C \nabla f \cdot d\mathbf{r}$
 (d) $\int_D \nabla f \cdot d\mathbf{r}$ (e) $\int_E \nabla f \cdot d\mathbf{r}$

- 10 Find the maximum value of the function $f(x, y) = 10x^{0.4}y^{0.6}$ subject to the constraint that $10x + 15y = 300$ where $x \geq 0$ and $y \geq 0$.

- 11 Find the surface integral of the vector field

$$\mathbf{F} = \langle -xy + yz, xy + 2y, -xz + yz \rangle$$

over the unit sphere, $x^2 + y^2 + z^2 = 1$ with an outward pointing normal vector.

- 12 (a) Let C be the curve in the plane described by the equation $x = \sin y$ for $0 \leq y \leq \pi$. Set up the integral $\int_C ds$ that computes the arc length of C . *You do not need to evaluate this integral.*

- (b) Let S be the surface in space obtained by rotating C around the y -axis. Set up the integral

$$\iint_S dS$$

that computes the surface area of S . *You do not need to evaluate this integral.*

- 13 Consider the triple iterated integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} \frac{y}{x+z} dz dx dy.$$

- (a) Convert the integral into cylindrical coordinates. *Do not evaluate the integral.*
 (b) Convert the integral into spherical coordinates. *Do not evaluate the integral.*

- 14 Consider the vector field in space given by $\mathbf{F}(x, y, z) = \langle x^3, y^3, 4z^3 \rangle$. Let S_1 be the disk in the xy -plane described by $x^2 + y^2 \leq 4$, $z = 0$; and let S_2 be the upper half of the paraboloid given by $z = 4 - x^2 - y^2$, $z \geq 0$. Both S_1 and S_2 are oriented upwards. Let E be the solid region enclosed by S_1 and S_2 .

- (a) Evaluate the flux integral $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

- (b) Calculate $\operatorname{div} \mathbf{F}$.

- (c) Calculate the iterated integral $\iiint_E \operatorname{div} \mathbf{F} dV$.

- (d) Evaluate the flux integral $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

Hint: Using parts (a) to (c), this point can be solved without performing any more integration. Use the Divergence Theorem and the fact that

$$\iint_{S_2} = \iint_{S_1} + \left(\iint_{S_2} - \iint_{S_1} \right).$$

- 15 If $\mathbf{F} = (\sin x^2)\mathbf{i} + (e^{y^2} + x^2)\mathbf{j} + (z^4 + 2x^2)\mathbf{k}$, use Stokes' Theorem to evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the triangle that goes from $(3, 0, 0)$ to $(0, 2, 0)$ to $(0, 0, 1)$ and back to $(3, 0, 0)$.

- 16 Let R be a region in the plane with boundary curve C , where C is a piecewise smooth curve oriented in the positive direction.

- (a) Use Green's Theorem to show that the area of R is given by the line integral

$$\frac{1}{2} \int_C -y dx + x dy.$$

- (b) Use part (a) to find the area of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$