1 True-False questions. Circle the correct letter. No justifications are required.

T F Any parameterized surface S is a graph of a function $f(x, y)$.

T F If $\mathbf{F}$ is a vector field of unit vectors defined in $1/2 \le x^2 + y^2 \le 2$ and $\mathbf{F}$ is tangent to the unit circle C, then $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is either equal to 2π or -2π.

T F If a curve C intersects a surface S at a right angle, then at the point of intersection, the tangent vector to the curve is parallel to the normal vector of the surface.

T F The curvature of the curve $\mathbf{r}(t) = \langle \cos(3t), \sin(6t) \rangle$ at the point $\mathbf{r}(0)$ is smaller than the curvature of the curve $\mathbf{r}(t) = \langle \cos(30t), \sin(60t) \rangle$ at the point $\mathbf{r}(0)$.

T F The following identity is true: $\int_0^3 \int_0^{2x} x^2 \, dy \, dx = \int_0^6 \int_{y/2}^3 x^2 \, dx \, dy$.

T F If $\mathbf{F} = \operatorname{curl} \mathbf{G}$, where $\mathbf{G} = \langle e^{e^x}, 5^x z^5, \sin y \rangle$, then $\operatorname{div} \mathbf{F}(x, y, z) > 0$ for all (x, y, z).

2 (a) Use vectors to show that the diagonals of a parallelogram are perpendicular if and only if all four sides of the parallelogram are equal in length (such a parallelogram is called a *rhombus*).

(b) What is the relationship between the cross product of the two diagonals of a rhombus, again considered as vectors, and the area of the rhombus?

3 Homer Simpson falls asleep at the controls of the Springfield nuclear power station and as a result the power plant releases radiation into the surrounding area. Suppose that the radiation intensity, measured in roentgens, is given by $f(x, y) = 2x + x^2 y + y^2$ where x and y are measured in miles.

(a) Suppose you are located at $(1, 1)$. In which direction should you move to decrease the radiation intensity as quickly as possible?

(b) Suppose you left your position at $(1, 1)$ on a bicycle in the direction of the vector $\langle 1, 2 \rangle$ at a speed of 10 miles per hour. What would be your rate of change of radiation intensity in roentgens per hour? Be sure to get the sign right; that is, would you be increasing or decreasing your exposure?

(c) Use linear approximation to estimate the level of radiation at your neighbor's house located at $(1.1, 1.2)$.

4 Consider the function $f(x, y) = x^2 + 2y^2 + x^2 y + 2004$. Find all the critical points of $f(x, y)$ and determine whether they are local maxima, local minima, or saddle points.

5 You're smuggling gold bars out of the country in small eggs. Find the volume of the rectangular bar with the largest volume that will fit inside an ellipsoid egg with equation

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

Assume that the rectangular bar has sides parallel to the three coordinate planes, and that x, y, z are all measured in inches.

6 You've just finished this exam and you're really thirsty. With Math 21a still on your mind, you go to the Paraboloid Bar, a famous non-alcoholic bar that serves all of its drinks in paraboloids. It's a good place to go, but you can never put your glass down a it will tip over immediately. Suppose that the glasses they use can be described by the equation $z = 2x^2 + 2y^2$, with $0 \le z \le 8$ and with x, y, and z measured in inches. Suppose that their glasses weigh 0.5 ounces per square inch of surface. Find the total weight of one glass.

7 You're still in the Paraboloid Bar (see previous question) as it was a long test and you need plenty of refreshment. You order one of their famous "Nine Pi" drinks. They pour 9π cubic inches of strange blue liquid into your paraboloid glass (described by the equation $z = 2x^2 + 2y^2$ with $0 \le z \le 8$ and x, y, and z measured in inches, as before). Will your cup run over? Find the height of the liquid in your glass, assuming that you're holding it so that the z-axis is straight up.

8 Evaluate the double integral $\int_0^1 \int_{x^2}^1 4x^3 \cos(y^3)\, dy\, dx$. Simplify your answer completely.

9 Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \big\langle -2\pi z^2 \sin(\pi x) e^y, 2z^2 \cos(\pi x) e^y, 4z \cos(\pi x) e^y \big\rangle$ and C is the upward-spiraling helix given by $\mathbf{r}(t) = \langle 2\cos(\pi t), 2\sin(\pi t), t\rangle$ for $0 \le t \le 6$.

10 Let $\mathbf{F} = \big\langle 2xy, x^2 z, 2z^2 \big\rangle$. Compute the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface of the rectangular box (oriented outward) bounded by the three coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$.

11
(a) Suppose that D is a simply-connected region in the xy-plane and that C is its boundary, with positive orientation. Give a careful explanation of why the area of D is given by the line integral $\frac{1}{2}\int_C x\, dy - y\, dx$.

(b) Use part (a) to compute the area of the region formed by the intersection of $x^2 + y^2 \le 2$ and $y \ge 1$.

12 Let C be the curve given by the intersection of the hyperbolic paraboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 9$. Let C be oriented counter-clockwise as one looks down the z-axis. Use Stokes' Theorem to calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field $\langle x^2 y, \frac{1}{3}x^3, xy\rangle$

13 Find the numerical value of the integral

$$\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} r^3 z^2\, dz\, dr\, d\theta$$

by changing it from an integral expressed in cylindrical coordinates to an integral expressed in spherical coordinates.

Note: While it is possible to calculate this integral in cylindrical coordinates, we want it done in spherical coordinates.

14 The intersection of the ellipsoid $4x^2 + y^2 + z^2 = 25$ with the plane $x = -2$ is a circle. Calculate the line integral of the vector field $\mathbf{F} = \langle xy - x, xz, x^2 y\rangle$ around this circle. Traverse this circle in the counterclockwise direction as viewed from the origin.

15 Consider the sphere of radius $\sqrt{2}$ centered about the origin which has equation $x^2 + y^2 + z^2 = 2$. Let S be the "drum" which is that part of the sphere with $-1 \le z \le 1$, together with its top and bottom.

(a) Find the surface area of S.

(b) Find the flux of the vector field $\mathbf{F} = \langle x, y, 0\rangle$ outward through the (spherical) side of the drum.

(c) Find the flux of the vector field $\mathbf{G} = \langle 0, 0, z\rangle$ outward through the entire surface of the drum.

(d) Based on your previous answers, find the flux of the vector field $\mathbf{H} = \langle x, y, z\rangle$ outward through the entire surface S of the drum.

(e) What is the volume of the drum?

Hint: You do not need to do any more integration for this part!