THIRD OLD FINAL EXAM

1 True-False questions. Circle the correct letter. No justifications are required.

T F The curve $\mathbf{r}(t) = (1-t)A + tB$, $t \in [0,1]$ connects the point A with the point B.

T F For every c, the function $u(x,t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

T F Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

T F The function $f(x,y,z) = x^2 - y^2 - z^2$ decreases in the direction $\langle 2,-2,-2\rangle/\sqrt{12}$ at the point $(1,1,1)$.

T F Assume $\mathbf{F}$ is a vector field satisfying $|\mathbf{F}(x,y,z)| \leq 1$ everywhere. For every curve C given by $\mathbf{r}(t)$, $0 \leq t \leq 1$, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is less or equal than the arc length of C.

T F Let $\mathbf{F}$ be a vector field which coincides with the unit normal vector $\mathbf{n}$ for each point on a curve C. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

T F If for two vector fields $\mathbf{F}$ and $\mathbf{G}$ one has $\operatorname{curl}\mathbf{F} = \operatorname{curl}\mathbf{G}$, then $\mathbf{F} = \mathbf{G} + \langle a,b,c\rangle$, where a, b, c are constants.

T F For every vector field $\mathbf{F}$ the identity $\operatorname{grad}(\operatorname{div}\mathbf{F}) = 0$ holds.

T F If $\operatorname{div}\mathbf{F}(x,y,z) = 0$ for all (x,y,z), then $\operatorname{curl}\mathbf{F} = \langle 0,0,0\rangle$ for all (x,y,z).

T F For every function $f(x,y,z)$, there exists a vector field $\mathbf{F}$ such that $\operatorname{div}\mathbf{F} = f$.

2 Indicate with a check in the column below "conservative" if a vector fields is conservative (that is if $\operatorname{curl}\mathbf{F}(x,y,z) = \langle 0,0,0\rangle$ for all points (x,y,z)). Similarly, mark the fields which are incompressible (that is if $\operatorname{div}\mathbf{F}(x,y,z) = 0$ for all (x,y,z)). No justifications are needed.

Vector Field	conservative $\operatorname{curl}\mathbf{F} = \mathbf{0}$	incompressible $\operatorname{div}\mathbf{F} = 0$
$\mathbf{F}(x,y,z) = \langle -5,5,3\rangle$		
$\mathbf{F}(x,y,z) = \langle x,y,z\rangle$		
$\mathbf{F}(x,y,z) = \langle -y,x,z\rangle$		
$\mathbf{F}(x,y,z) = \langle x^2+y^2, xyz, x-y+z\rangle$		
$\mathbf{F}(x,y,z) = \langle x-2yz, y-2zx, z-2xy\rangle$		

3 Let E be a parallelogram in three dimensional space defined by two vectors $\mathbf{u}$ and $\mathbf{v}$.

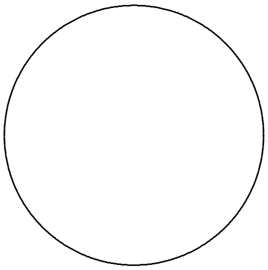
(a) Express the diagonals of the parallelogram as vectors in terms of $\mathbf{u}$ and $\mathbf{v}$.

(b) What is the relation between the length of the cross product of the diagonals and the area of the parallelogram?

(c) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

4 Find the volume of the wedge shaped solid that lies above the xy-plane, below the plane $z = x$, and within the cylinder $x^2 + y^2 = 4$.

5 Match the equations with the objects. No justifications are needed.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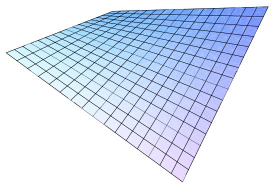
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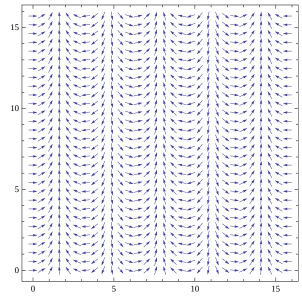
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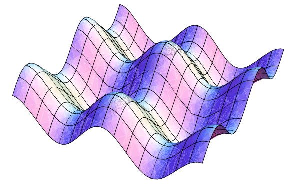
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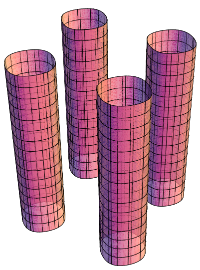
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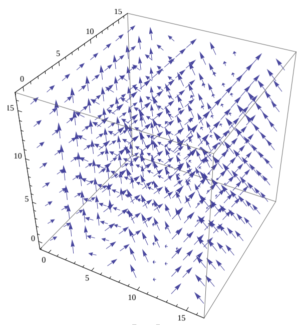
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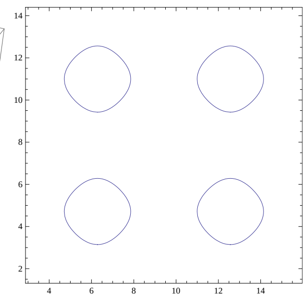
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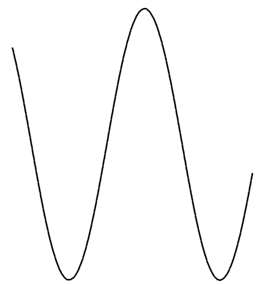

VIII

Equations:

(a) $g(x, y, z) = \cos(x) + \sin(y) = 1$

(b) $y = \cos(x) - \sin(x)$

(c) $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$

(d) $\mathbf{r}(u, v) = \langle \cos(u), \sin(v), \cos(u)\sin(v) \rangle$

(e) $\mathbf{F}(x, y, z) = \langle \cos(x), \sin(x), 1 \rangle$

(f) $z = f(x, y) = \cos(x) + \sin(y)$

(g) $g(x, y) = \cos(x) - \sin(y) = 1$

(h) $\mathbf{F}(x, y) = \langle \cos(x), \sin(x) \rangle$

6 Consider the surface parameterized by

$$x = uv\cos v \qquad y = uv\sin v \qquad z = v,$$

where $0 \le u \le 1$ and $0 \le v \le \pi$ Which of the following integrals is an integral for the surface area of this surface?

(a) $\displaystyle\int_0^\pi \int_0^1 \sqrt{u^2 + u^2v^4}\, du\, dv$

(b) $\displaystyle\int_0^\pi \int_0^1 \sqrt{v^2 + u^4v^2}\, du\, dv$

(c) $\displaystyle\int_0^\pi \int_0^1 \sqrt{v^2 + u^2v^4}\, du\, dv$

(d) $\displaystyle\int_0^\pi \int_0^1 \sqrt{u^2 + u^4v^4}\, du\, dv$

(e) None of the above.

7 Let the curve C be parameterized by $\mathbf{r}(t) = \langle t, \sin t, t^2 \cos t\rangle$ for $0 \le t \le \pi$. Let $f(x, y, z) = z^2 e^{x+2y} + x^2$ and $\mathbf{F} = \nabla f$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

8 Evaluate the line integral of the vector field $\mathbf{F}(x, y) = \langle y^2, x^2\rangle$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0, 0)$, $(1, 0)$ and $(1, 1)$ in two ways:

(a) By evaluating the three line integrals.

(b) Using Green's theorem.

9 Evaluate $\displaystyle\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8}\, dx\, dy$.

10 Match each of the following iterated integrals with its domain of integration.

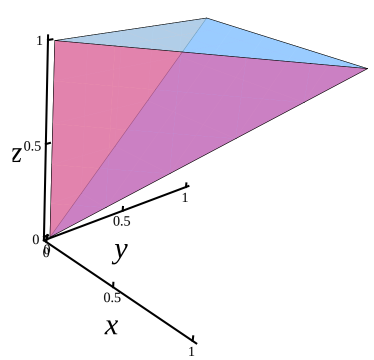
(a) $\displaystyle\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$

(b) $\displaystyle\int_0^1 \int_0^y \int_y^1 f(x, y, z)\, dz\, dx\, dy$

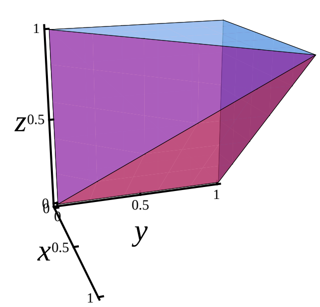
(c) $\displaystyle\int_0^1 \int_y^1 \int_0^x f(x, y, z)\, dz\, dx\, dy$

(d) $\displaystyle\int_0^1 \int_0^y \int_x^1 f(x, y, z)\, dz\, dx\, dy$

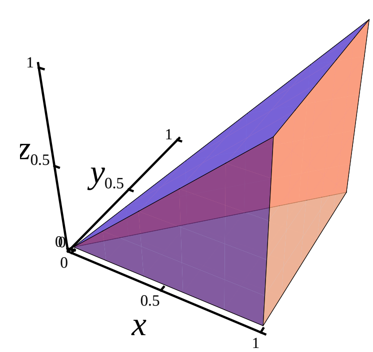
(e) $\displaystyle\int_0^1 \int_0^y \int_x^y f(x, y, z)\, dz\, dx\, dy$



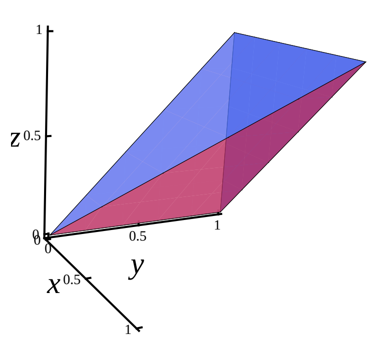
Region (i)



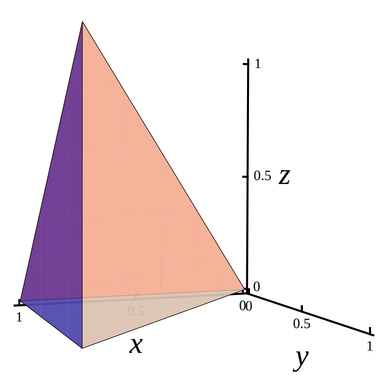
Region (ii)



Region (iii)



Region (iv)



Region (v)

11 Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

12 (a) Find all the critical points of the function $f(x, y) = -(x^4 - 8x^2 + y^2 + 1)$.

(b) Classify the critical points.

(c) Locate the local and absolute maxima of f.

(d) Find the equation for the tangent plane to the graph of f at each absolute maximum.

13 Use Stokes' theorem to evaluate the line integral of $\mathbf{F}(x, y, z) = \langle -y^3, x^3, -z^3 \rangle$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 1 - \cos(t) - \sin(t) \rangle$ with $0 \le t \le 2\pi$.

14 Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $\{(x, y) \ : \ x^2 + y^2 \le 1\}$ in the xy-plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field

$$\mathbf{F} = \left\langle -4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \right\rangle$$

through S using the divergence theorem.