Math 21a

Fourth Old Final Exam

Spring, 2009

1 True-False questions. Circle the correct letter. No justifications are required.

T F The projection vector $\text{proj}_{\mathbf{v}}(\mathbf{w})$ is parallel to $\mathbf{w}$.

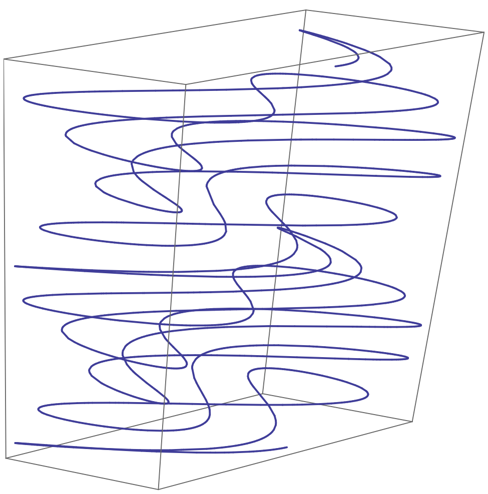
T F If the directional derivatives $D_{\mathbf{v}}f(1,1)$ and $D_{\mathbf{w}}f(1,1)$ are both 0 for $\mathbf{v} = \langle 1, \frac{1}{2} \rangle$ and $\mathbf{w} = \langle 1, -\frac{1}{2} \rangle$, then $(1,1)$ is a critical point.

T F The linearization $L(x,y)$ of $f(x,y) = x + y + 4$ at $(0,0)$ satisfies $L(x,y) = f(x,y)$.

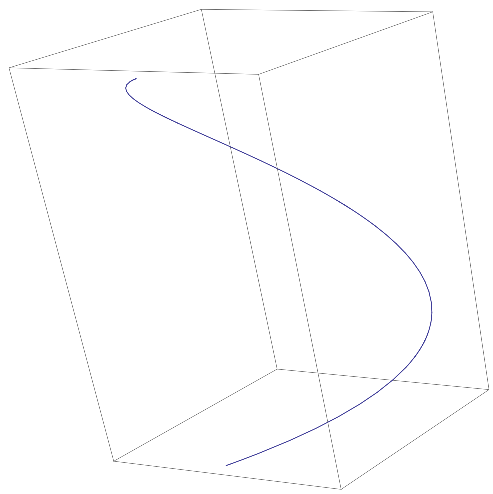
T F The integral $\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2 \phi \; d\phi \; d\theta \; d\rho$ is equal to the volume of the unit ball.

T F The integral $\int_0^x \int_y^1 f(x,y) \; dx \; dy$ represents a double integral over a bounded region in the plane.

2 Match the equations with the space curves. No justifications are needed.



I



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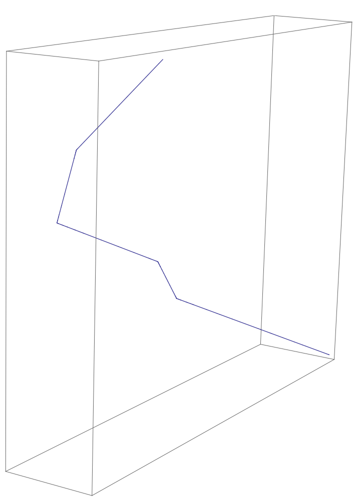

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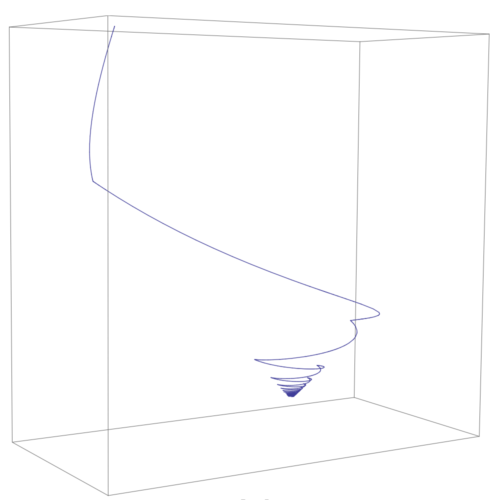


IV

Equations:

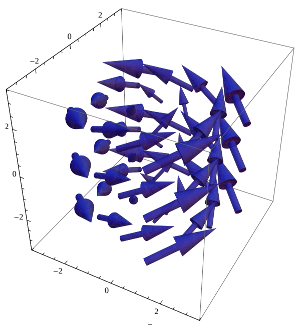
(a) $\mathbf{r}(t) = \langle t^2, t^3 - t, t \rangle$

(b) $\mathbf{r}(t) = \langle |1 - |t||, |t - |t - 1||, t \rangle$

(c) $\mathbf{r}(t) = \langle 2\sin(5t), \cos(11t), t \rangle$

(d) $\mathbf{r}(t) = \langle t\sin(1/t), t|\cos(1/t)|, t \rangle$

3 Match the equations with the objects. No justifications are needed.



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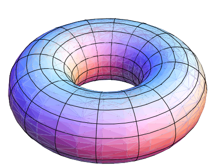

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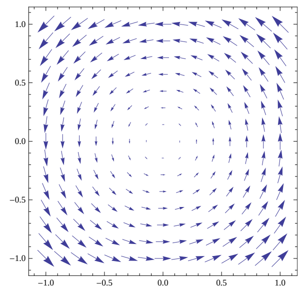

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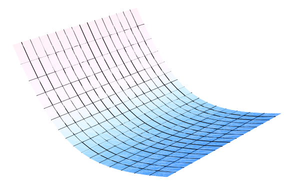

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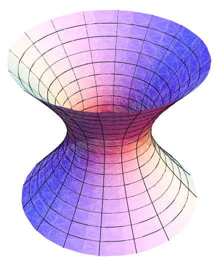

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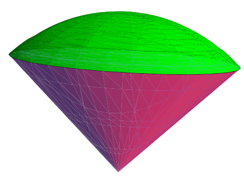

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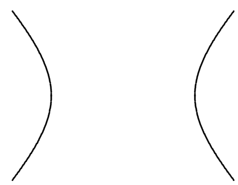
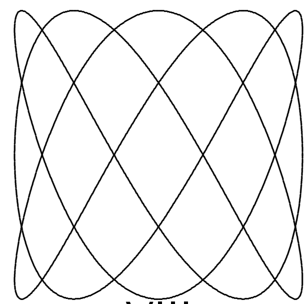


VII



VIII

Equations:

(a) $\mathbf{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s)\rangle$

(b) $x^2+y^2+z^2 \leq 1$, $x^2+y^2 \leq z^2$, $z \geq 0$

(c) $\mathbf{r}(t) = \langle \cos(3t), \sin(5t)\rangle$

(d) $x^2+y^2-z^2 = 1$

(e) $\mathbf{F}(x,y,z) = \langle -y, x, 1\rangle$

(f) $z = f(x,y) = x^2 - y$

(g) $g(x,y) = x^2 - y^2 = 1$

(h) $\mathbf{F}(x,y) = \langle -y, x\rangle$

4 (a) Find an equation for the plane Π passing through the points $\mathbf{r}(0)$, $\mathbf{r}(1)$, $\mathbf{r}(2)$, where $\mathbf{r}(t) = \langle t^2, t^4, t\rangle$.

(b) Find the distance between the point $\mathbf{r}(-1)$ and the plane Π found in part (a).

5 A vector field $\mathbf{F}(x,y)$ in the plane is given by $\mathbf{F}(x,y) = \langle x^2+5, y^2-1\rangle$. Find all the critical points of $|\mathbf{F}(x,y)|$ and classify them. At which point or points is $|\mathbf{F}(x,y)|$ minimal?
Hint: Extremize $f(x,y) = |\mathbf{F}(x,y)|^2$.

6 A house is situated at the point $(0,0)$ in the middle of a mountainous region. The altitude at each point (x,y) is given by the equation $f(x,y) = 4x^2y + y^3$. There is a pathway in the shape of an ellipse around the house, on which the (x,y) coordinates satisfy $2x^2+y^2 = 6$. Find the highest and lowest points in the closed region bounded by the path.

7 (a) Where does the tangent plane at $(1,1,1)$ to the surface $z = e^{x-y}$ intersect the z axis?

(b) Estimate $f(x,y,z) = 1 + \log(1+x+2y+z) + 2\sqrt{1+z}$ at the point $(0.02, -0.001, 0.01)$.

(c) $f(x,y,z) = 0$ defines z as a function $g(x,y)$ of x and y. Find the partial derivative $g_x(x,y)$ at the point $(x,y) = (0,0)$.

8 For each of the following quantities, set up a double or triple integral using any coordinate system you like. You do not have to evaluate the integrals, but the bounds of each single integral must be specified explicitly.

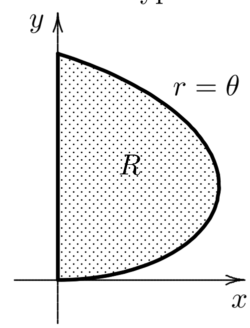
(a) The volume of the tetrahedron with vertices $(0,0,0)$, $(3,0,0)$, $(0,3,0)$ and $(0,0,3)$.

(b) The surface area of the piece of the paraboloid $z = x^2 + y^2$ lying in the region $x^2 + y^2 \leq 1$.

(c) The volume of the solid bounded by the planes $z = -1$, $z = 1$ and the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

9 A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta$ for $\theta \in [0, \pi/2]$. Find the double integral

$$\iint_R \frac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}(\pi/2 - \sqrt{x^2+y^2})}\, dx\, dy.$$



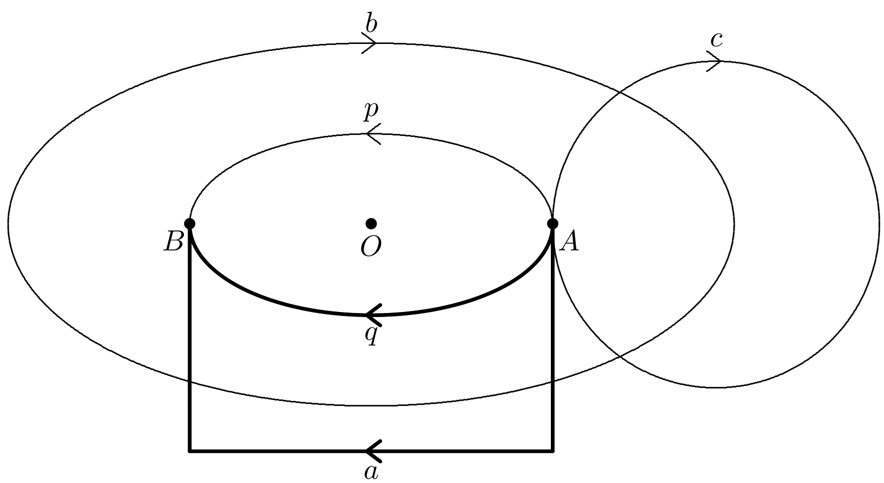
10 A car drives up a freeway ramp which is parametrized by $\mathbf{r}(t) = \langle \cos(t), 2\sin(t), t \rangle$, $0 \leq t \leq 3\pi$.

(a) Set up an integral which gives the length of the ramp. You do need not need to evaluate it.

(b) Find the unit tangent vector $\mathbf{T}$ to the curve at the point where $t = 0$.

(c) Suppose the wind pattern in the area is such that the wind exerts a force $\mathbf{F} = \langle 4x^2, y, 0 \rangle$ on the car at the position (x, y, z). What is the total work done by the car against the wind as it drives up the ramp?

11 Suppose $\mathbf{F}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $\mathbf{O} = \langle 0, 0 \rangle$. Suppose the line integral of $\mathbf{F}$ along the path p from A to B is 5 and the line integral of $\mathbf{F}$ along the path q from A to B is -4. Find the line integral of $\mathbf{F}$ along the following three paths:



(a) The path a from A to B going clockwise below the origin.

(b) The closed path b encircling the origin in a clockwise direction.

(c) The closed path c starting at A and ending in A without encircling the origin.

12 Let S be the surface which bounds the region enclosed by the paraboloid $z = x^2 + y^2 - 1$ and the xy-plane. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle x + e^{\sin(z)}, z, -y \rangle$.

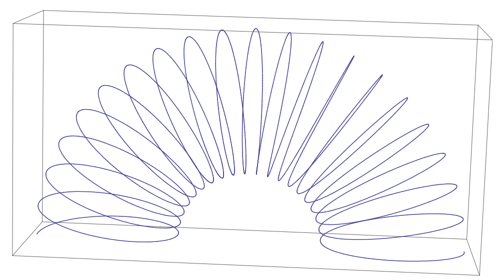
(a) Find the flux of $\mathbf{F}$ through the surface S.

(b) Find the flux of $\mathbf{F}$ through the part of the surface S that belongs to the paraboloid, oriented so that the normal vector points downwards.

13 Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle 4z + \cos(\cos x), y^2, x + 2y \rangle$.

(a) Let C be the curve given by the parameterization $\mathbf{r}(t) = \langle \cos t, 0, \sin t \rangle$, for $0 \le t \le 2\pi$. Find the line integral of $\mathbf{F}$ along $\mathbf{C}$.

(b) Let S be the hemisphere of the unit sphere defined by $y \le 0$. Find the flux of the curl of $\mathbf{F}$ out of S. In other words, find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. For part (b), the surface S is oriented so that the normal vector has a positive y-component.



14 Find the line integral of the vector field

$$\mathbf{F}(x, y, z) = \left\langle \cos(x + z), 2yze^{y^2 z}, \cos(x + z) + y^2 e^{y^2 z} \right\rangle$$

along the *Slinky* curve

$$\mathbf{r}(t) = \left\langle \sin(40t), (2 + \cos(40t)) \sin(t), (2 + \cos(40t)) \cos(t) \right\rangle$$

with $0 \le t \le \pi$.