This should really be titled "Answers" rather than "Solutions." I've tried to comment a little here and there, especially on the problems that weren't actually discussed in the review session.

1 Let S be the piece of the paraboloid $z = 1 - x^2 - y^2$ where $z \geq 0$. Let $\mathbf{n}$ denote the normal of S which points in the $+z$ direction at $(0, 0, 1)$. Let $\mathbf{F}$ denote the vector field

$$\mathbf{F} = \big\langle x + y\sin(z^2), y + x\sin(z^2), 1 - 2z\big\rangle.$$

(a) By parameterizing S, write down a double integral that computes the flux of $\mathbf{F}$ through S in the direction $\mathbf{n}$.

Solution: The parameterization is simply $\mathbf{r}(x, y) = \langle x, y, 1 - x^2 - y^2\rangle$, so $\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1\rangle$ is the normal (properly oriented!). The flux is then

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \big\langle x+y\sin((1-x^2-y^2)^2), y+x\sin((1-x^2-y^2)^2), 1-2(1-x^2-y^2)\big\rangle\cdot\langle 2x, 2y, 1\rangle\, dx\, dy$$

or, in polar coordinates,

$$\int_0^{2\pi}\int_0^1 \big\langle r\cos(\theta) + r\sin(\theta)\sin((1-r^2)^2), r\sin(\theta) + r\cos(\theta)\sin((1-r^2)^2), 1 - 2(1-r^2)\big\rangle \cdot \langle 2r\cos(\theta), 2r\sin(\theta), 1\rangle\, r\, dr\, d\theta.$$

I don't really want to compute either of those.

(b) Find a surface $\tilde{S}$ which is not S such that the fluxes of $\mathbf{F}$ through S and $\tilde{S}$ have the same absolute value.

Solution: We're going to choose $\tilde{S}$ to that the union of S and $\tilde{S}$ together bounds a solid E. The simplest such $\tilde{S}$ is the unit disk in the xy-plane. If we orient $\tilde{S}$ with the downward-pointing normal $\mathbf{n} = -\mathbf{k} = \langle 0, 0, -1\rangle$, then $S \cup \tilde{S}$ is oriented with the normal that points outward from E. This means we can apply the divergence theorem, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} + \iint_{\tilde{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV.$$

Notice that $\operatorname{div}\mathbf{F} = 0$, so this equation means that

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_{\tilde{S}} \mathbf{F}\cdot d\mathbf{S}.$$

In particular, $\tilde{S}$ is a surface with the requested properties.

(c) What is the value of the integral that you wrote down for part (a)?

Solution: I'm *not* going to calculate the integral given in part (a). It's just too ugly. Instead, I'll use our result from part (b) and integrate over the disk $\tilde{S}$. Remember that $d\mathbf{S} = \mathbf{n}\, dS$, and here $\mathbf{n} = -\mathbf{k}$ and $dS = dA$ (since the surface lies in the xy-plane. In polar coordinates, then, the flux across S is

$$\begin{aligned} -\iint_{\tilde{S}} \mathbf{F}\cdot d\mathbf{S} &= -\int_0^{2\pi}\int_0^1 \mathbf{F}\cdot\langle 0, 0, -1\rangle\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 (1 - 2z)\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 r\, dr\, d\theta \qquad \text{since } z = 0 \text{ on } \tilde{S}. \end{aligned}$$

This last integral is simply the area of the unit disk in the xy-plane, which is $\pi(1)^2 = \pi$. Thus the flux of $\mathbf{F}$ across S is π.

2 Let $\mathbf{F} = \langle x + xz, y - yz, z^2 \rangle$. Here are two surfaces in $\mathbf{R}^3$ with the same boundary:

$$A: \quad z = \sqrt{1 - x^2 - y^2}, \qquad \text{with } x^2 + y^2 \leq 1$$
$$B: \quad z = 2\sqrt{1 - x^2 - y^2}, \qquad \text{with } x^2 + y^2 \leq 1.$$

(a) Which of the surfaces has the greatest flux of $\mathbf{F}$ through it? For both surfaces, use the normal which has a positive dot product with $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Solution: Each of A and B are half of an ellipsoid: A is a hemisphere and B shares the boundary of A but has been "stretched" vertically by a factor of 2. Since A and B share a boundary (the unit circle in the xy-plane), they bound a solid E. But A is oriented incorrectly to be the boundary of this solid; really the boundary of E is $B \cup (-A)$. Thus the divergence theorem says that

$$\iint_B \mathbf{F} \cdot d\mathbf{S} - \iint_A \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

But $\operatorname{div} \mathbf{F} = 2 + 2z$, which is always at least 2 in E. Thus the triple integral is positive, so $\iint_B \mathbf{F} \cdot d\mathbf{S} > \iint_A \mathbf{F} \cdot d\mathbf{S}$.

(b) Which of the surfaces has the greatest flux of $\operatorname{curl} \mathbf{F}$ through it? For both surfaces, use the same normal as in part (a).

Solution: If I repeat the previous solution with $\operatorname{curl} \mathbf{F}$ in the place of $\mathbf{F}$, I get

$$\iint_B \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} - \iint_A \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \operatorname{curl} \mathbf{F} \, dV = 0.$$

(The last equality is because $\operatorname{div} \operatorname{curl} \mathbf{F}$ is always zero.) Thus $\iint_B \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_A \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

(c) Compute the flux of $\operatorname{curl} \mathbf{F}$ through each surface.

Solution: From part (b), we can compute using either surface A or surface B or, for that matter, any other surface sharing the same boundary as A and B. So let's use S, the unit disk in the xy-plane. Since

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + xz & y - yz & z^2 \end{vmatrix} = \langle y, x, 0 \rangle.$$

and the normal to S is either $\pm\mathbf{k} = \langle 0, 0, \pm 1 \rangle$, we get $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = 0$. Thus the flux is zero.

(For those of you are still confused, remember that the vector surface area element is $d\mathbf{S} = \mathbf{n} \, dS$, the normal $\mathbf{n}$ times the scalar surface area element. So $\mathbf{F} \cdot \mathbf{n} = 0$ means $\mathbf{F} \cdot d\mathbf{S} = 0$ as well.)

3 Find the volume of the solid bounded below by $z = 0$, above by $z^2 = 3x^2 + 3y^2$, and on the side by $x^2 + y^2 + z^2 = 4$.

Solution: This solid can be described in spherical coordinates as

$$\left\{ (\rho, \phi, \theta) \ : \ 0 \leq \rho \leq 2, \ \tfrac{\pi}{6} \leq \phi \leq \tfrac{\pi}{6}, \ 0 \leq \theta \leq 2\pi \right\}.$$

The tricky part is the lower bound on ϕ. This is where $z^2 = 3(x^2 + y^2)$, or $z = \sqrt{3}r$ (in cylindrical coordinates). Since $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$, this means $\cos(\phi) = \sqrt{3}\sin(\phi)$, or $\tan(\phi) = \frac{1}{\sqrt{3}}$. This is $\phi = \frac{\pi}{6}$.

Thus this volume is

$$V = \iiint_E 1\ dV = \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^2 \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$$

$$= 2\pi \cdot \Big[-\cos(\phi) \Big]_{\pi/6}^{\pi/2} \cdot \frac{2^3}{3} = \frac{16\pi}{3} \cdot \frac{\sqrt{3}}{2} = \frac{8\pi}{\sqrt{3}}.$$

4 Water is flowing down a vertical cylindrical pipe of radius 2 inches. The velocity vector field of the water at the outlet of the pipe is given by $\mathbf{v} = (r^2 - 4)\mathbf{k}$, where r is the distance in inches from the center of the pipe. How much water flows out of the bottom of the pipe in 3 seconds?

Solution: The flux integral of $\mathbf{v}$ will give us the rate (in cubic inches per second, presumably) of flow out of the pipe. The outlet of the pipe is the disk of radius 2 inches in the xy-plane. We parameterize the outlet of the pipe by $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle$, so $\mathbf{v} = \langle 0, 0, r^2 - 4\rangle$ in these coordinates. We compute

$$\mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \end{vmatrix} = \langle 0, 0, -r\rangle = -r\ \mathbf{k}.$$

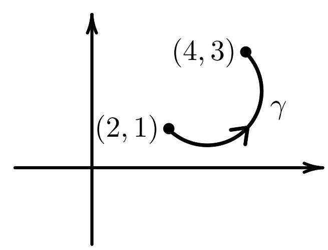
(Notice that this is the proper orientation, as we're interested in how much water flows *down* through the pipe.) Thus the flux integral over this surface S is

$$\iint_S \mathbf{v} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 (r^2 - 4)(-r)\ dr\ d\theta = \int_0^{2\pi} 4\ d\theta = 8\pi.$$

Thus the water flows out at 8π in^3/sec, so in 3 seconds a total of 24π cubic inches of water flows out.

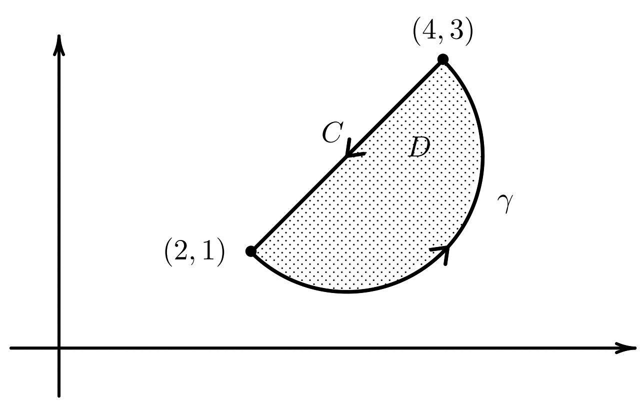
5 Calculate the work integral

$$\int_\gamma (x + y)\ dx + (3x - 2y)\ dy$$



for the curve γ shown, a semicircle from the point $(2, 1)$ to the point $(4, 3)$.

Solution: Here's the picture re-drawn, with another curve added and a region shaded:



Now $\gamma \cup C$ is the (oriented) boundary of D, so Green's theorem applies:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA = \int_\gamma P\ dx + Q\ dy + \int_C P\ dx + Q\ dy.$$

Here $P = x + y$ and $Q = 3x - 2y$, so $Q_x - P_y = 3 - 1 = 2$, so

$$\int_\gamma (x+y)\,dx + (3x-2y)\,dy = \iint_D 2\,dA - \int_C (x+y)\,dx + (3x-2y)\,dy.$$

Two quick things to notice: First, the double integral is simply twice the area of the half disk (whose radius is $\sqrt{2}$), or simply $2 \cdot \frac{1}{2}\pi(\sqrt{2})^2 = 2\pi$. Second, the curve C is much simpler to parameterize than the half-circle γ. One simple parameterization is $\mathbf{r}(t) = \langle 4, 3\rangle + t\left(\langle 2, 1\rangle - \langle 4, 3\rangle\right)$, or $\mathbf{r}(t) = \langle x, y\rangle = \langle 4 - 2t, 3 - 2t\rangle$ with $0 \le t \le 1$. Thus $dx = -2\,dt = dy$, so

$$\begin{aligned}\int_C (x+y)\,dx + (3x-2y)\,dy &= \int_0^1 (4 - 2t + 3 - 2t)\,(-2\,dt) + [3(4-2t) - 2(3-2t)]\,(-2\,dt) \\ &= \int_0^1 (12t - 30)\,dt = \Big[6t^2 - 30t\Big]_0^1 = -24.\end{aligned}$$

Thus $\displaystyle\int_\gamma (x+y)\,dx + (3x-2y)\,dy = 2\pi + 24.$

6 Compute the flux of the vector field $\mathbf{F}(x, y, z) = \left\langle e^{y^2+z^2}, y^2 + z^2, e^{x^2+y^2}\right\rangle$ across a portion of the cone with equation $4(x^2 + y^2) = 9z^2$ lying between $z = 0$ and $z = 2$ oriented with a downward normal.

Solution: By the looks of the vector field, I don't think we want to integrate this directly. Let's try the divergence theorem, since $\operatorname{div}\mathbf{F} = 0 + 2y + 0 = 2y$. We can make the surface closed by adding the surface S_2, the disk $x^2 + y^2 \le 4$ in the plane $z = 2$, oriented with the upward (and outward) pointing normal. Then the divergence theorem says that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV,$$

where E is the solid cone between $z = 0$ and $z = 2$

We'll start with the flux integral over S_2. This has parameterization $\mathbf{r}(x, y) = \langle x, y, 2\rangle$ (with (x, y) lying in the disk D of radius 3), so $d\mathbf{S} = \mathbf{k}\,dx\,dy$. Thus

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \langle e^{y^2+2^2}, y^2 + 2^2, e^{x^2+y^2}\rangle \cdot \langle 0, 0, 1\rangle\,dx\,dy \\ &= \iint_D e^{x^2+y^2}\,dx\,dy \\ &= \int_0^{2\pi} \int_0^3 e^{r^2}\,r\,dr\,d\theta \\ &= \pi\left(e^9 - 1\right).\end{aligned}$$

The triple integral is the integral over the region with $\frac{2}{3}r \le z \le 2$, so we use cylindrical coordinates:

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E -2y\,dV \\ &= -2\int_0^{2\pi} \int_0^2 \int_{2r/3}^2 r\sin(\theta)\,r\,dz\,dr\,d\theta \\ &= 0.\end{aligned}$$

This makes sense, since this solid cone E is symmetric with respect to y.

The grand conclusion is that the flux through S is the opposite of the flux through S_2, thus our flux through S is $\pi\left(1 - e^9\right)$.

7 Write down, but do not evaluate, a double integral that computes the surface area of the part of the surface $x^4 + 2x^2y^2 + y^4 + z^4 = 16$ with $x \geq 0$.

Solution: Notice that the surface equation can be re-written as $z^4 + (x^2 + y^2)^2 = 16$. Thus $x^2 + y^2 = \sqrt{16 - z^4}$; that is, for fixed z, the traces are circles of radius $\sqrt[4]{16 - z^4}$. Thus we'll parameterize this

$$\mathbf{r}(\theta, z) = \left\langle \sqrt[4]{16 - z^4}\cos(theta), \sqrt[4]{16 - z^4}\sin(theta), z \right\rangle,$$

where $-2 \leq z \leq 2$ (since $2 = \sqrt[4]{16}$) and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (since $x \geq 0$). Then

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt[4]{16 - z^4}\sin(\theta) & \sqrt[4]{16 - z^4}\cos(\theta) & 0 \\ -\frac{z^3}{(16-z^4)^{3/4}}\cos(\theta) & -\frac{z^3}{(16-z^4)^{3/4}}\sin(\theta) & 1 \end{vmatrix}$$

$$= \left\langle \sqrt[4]{16 - z^4}\cos(\theta), \sqrt[4]{16 - z^4}\sin(\theta), \frac{z^3}{\sqrt{16 - z^4}} \right\rangle.$$

This has length

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\sqrt{16 - z^4}\left(\cos^2(\theta) + \sin^2(\theta)\right) + \frac{z^6}{16 - z^4}} = \sqrt{\frac{(16 - z^4)^{3/2} + z^6}{16 - z^4}}$$

Thus the surface area of the required surface is

$$\int_{-\pi/2}^{\pi/2} \int_{-2}^{2} \sqrt{\frac{(16 - z^4)^{3/2} + z^6}{16 - z^4}}\, dz\, d\theta.$$

8 Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = \langle xy^2, xy^2, xy^2 \rangle$. Let D be the portion of the solid ball $x^2 + y^2 + z^2 \leq 9$ which lies in the first octant (that is, $x \geq 0$, $y \geq 0$, and $z \geq 0$). Set up, but do not evaluate, a triple integral in spherical coordinates which gives the flux of $\mathbf{F}$ out of the boundary of the region D.

Solution: The divergence theorem tells us that the flux of $\mathbf{F}$ out of the boundary of the region D is the triple integral of $\operatorname{div}\mathbf{F}$ over D. In spherical coordinates this triple integral is fairly simple:

$$\text{flux} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \operatorname{div}\mathbf{F}\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Since $\operatorname{div}\mathbf{F} = y^2 + 2xy + 0$, we can write this divergence in spherical coordinates as $\rho^2\sin^2(\phi)\sin^2(\theta) + 2\rho^2\sin^2(\phi)\sin(\theta)\cos(\theta)$. Thus the final integral for the flux is

$$\text{flux} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^4 \sin^3(\phi)\left[\sin^2(\theta) + 2\sin(\theta)\cos(\theta)\right] d\rho\, d\phi\, d\theta.$$

9 Let $f(x, y, z)$ be a potential function for a conservative vector field $\mathbf{F}$; i.e., $\mathbf{F} = \nabla f$. Consider a level surface M for the function f, where M is given by $f(x, y, z) = k$, for some constant k. If C is a curve (not necessarily closed) on M, explain why $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Solution: Recall that the Fundamental Theorem of Line Integrals says that

$$\int_C \nabla f \cdot d\mathbf{r} = f(\text{ end point of } C\ ) - f(\text{ start point of } C\ ).$$

The fact that C starts and ends on a level surface is the key. On this level surface, the function f is always equal to some number, say k. Then $f($ end point of $C\ ) = k$ and $f($ start point of $C\ ) = k$, since both points lie on this level surface. Thus the line integral is zero.

10 Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle -e^y \sin x, e^y \cos x \right\rangle$ and C is the curve given by $\mathbf{r} = \langle \pi - \pi\cos t, \pi \sin t\rangle$ with $0 \le t \le \pi$.

Solution: If I try to compute this directly, I very quickly realize that this is computationally difficult. I have a couple options: Green's theorem (although the curve C isn't closed, maybe I can find a simpler curve?) or the fundamental theorem of line integrals.

To use the fundamental theorem, I need to see if $\mathbf{F}$ is conservative. Writing $\mathbf{F} = \langle P, Q\rangle$, we see quickly that, indeed, $Q_x = P_y = -e^y \cos(x)$. Thus $\mathbf{F}$ is conservative.

To use this I need a potential function f with $\mathbf{F} = \langle f_x, f_y\rangle$. Thus

$$\begin{aligned} f_x &= -e^y \sin x \\ f_y &= e^y \cos x, \end{aligned}$$

so we can find that

$$f_x = \int -e^y \sin x \, dx = e^y \cos x + g(y),$$

where $g(y)$ is some function of y. We check that $f_y = e^y \cos x + g'(y) = e^y \cos x$, so $g'(y) = 0$. We'll take $f(x, y) = e^y \cos(x)$ (that is, $g(y) = 0$), as this is one possible potential function.

Finally, we thus have

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &= f(\text{ end point of } C\ ) - f(\text{ start point of } C\ ) \\ &= f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(2\pi, 0) - f(0,0) \\ &= e^0 \cos(2\pi) - e^0 \cos(0) = 0. \end{aligned}$$

11 Use Green's Theorem and the vector field $\mathbf{F} = \langle 0, x^3 y\rangle$ to compute the integral $\iint\limits_R 3x^2 y \; dA$, where R is the region inside the ellipse $x^2 + \frac{y^2}{4} = 1$. For the boundary of the ellipse note that $\cos^2 t + \frac{(2\sin t)^2}{4} = 1$.

Solution: Green's theorem says that

$$\iint\limits_R 3x^2 y \, dA = \oint\limits_C \mathbf{F}\cdot d\mathbf{r} = \oint\limits_C x^3 y \, dy.$$

Using the hint, we parameterize the ellipse C by $\mathbf{r}(t) = \langle x, y\rangle = \langle \cos t, 2\sin t\rangle$. Then $d\mathbf{r} = \mathbf{r}'(t)\, dt = \langle -\sin t, 2\cos t\rangle\, dt$, and so

$$\iint\limits_R 3x^2 y \, dA = \oint\limits_C x^3 y \, dy = \int_0^{2\pi} (\cos^3 t)\,(2\sin t)\, 2\cos t \, dt = 4\int_0^{2\pi} \cos^4 t \sin t \, dt.$$

Using the substitution $u = \cos t$ (so $du = -\sin t \, dt$), we get that the integral is zero.

We remark that this makes sense, since the integrand $3x^2 y$ is symmetric with respect to y in the original region R.

12 Let S be the surface given by $z = x^2 - y^2$. Let C be the curve on the surface S given by $x^2 + y^2 = 1$ and oriented counterclockwise as one looks down the z- axis. Use Stokes' Theorem to calculate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^2 + z^2, y, z\rangle$.

Solution: Let T be the part of the surface S that lies inside the curve C, oriented with the upward-pointing normal to be compatible with C. Then Stokes' theorem says that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_T \operatorname{curl} \mathbf{F}\cdot d\mathbf{S}.$$

Here T is parameterized by $\mathbf{r}(x,y) = \langle x, y, x^2 - y^2 \rangle$, from which we get

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle .$$

In this system, $\operatorname{curl} \mathbf{F} = \langle 0, 2z, 0 \rangle = \langle 0, 2(x^2 - y^2), 0 \rangle$, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_T \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -4y(x^2 - y^2)\, dy\, dx.$$

We compute this integral fairly easily:

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (4y^3 - 4x^2 y)\, dy\, dx \\ &= \int_{-1}^{1} \left[y^4 - 2x^2 y^2 \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} 0\, dx = 0. \end{aligned}$$

13 In appropriate units, the charge density $\sigma(x, y, z)$ in a region in space is given by $\sigma = \nabla \cdot \mathbf{E} = \operatorname{div} \mathbf{E}$, where $\mathbf{E}$ is the electrical field. Consider the unit cube given by $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. What is the total charge in this cube if

$$\mathbf{E} = \langle x(1-x)\ln(1+xyz), y(1-y)\tan(xyz), z(1-z)e^{xyz} \rangle.$$

You are being asked to integrate the charge density σ over the cube. You do not need to know any physics to do this problem.

Solution: We're asked to find $\iiint_R \operatorname{div} \mathbf{E}\ dV$, where R is the unit cube. By the Divergence theorem, this is the flux of $\mathbf{E}$ through the six faces of the unit cube. Each of these six fluxes is zero, so the total charge density is zero.

Why is each flux zero? Let's look at the flux through the two faces $x = 0$ and $x = 1$. On these faces the normal is either $\mathbf{i} = \langle 1, 0, 0 \rangle$ or $-\mathbf{i} = \langle -1, 0, 0 \rangle$. Moreover the first component of $\mathbf{E}$ is zero on these faces, since either $x = 0$ or $1 - x = 0$. Thus $\mathbf{E} \cdot \mathbf{n} = 0$, and so $\mathbf{E} \cdot d\mathbf{S} = 0$. This means that the flux $\iint_{\text{face}} \mathbf{E} \cdot d\mathbf{S} = 0$ over each face $x = 0$ and $x = 1$.

The flux thought the other four faces are similarly zero.