Some Solutions to Review Session 2 Problems

5. Suppose $g(x, y, z)$ is a function for which $g(2, 1, 3) = 6$, $g_x(2, 1, 3) = 5$, $g_y(2, 1, 3) = -2$, and $g_z(2, 1, 3) = 4$.

   (a) *Let S be the level surface of g that passes through the point $(2, 1, 3)$. Find the equation of the plane tangent to S at $(2, 1, 3)$.*

   Solution. We know that ∇g is perpendicular to level surfaces of g, so $\nabla g(2, 1, 3)$ is a normal vector for the tangent plane. From the given information, $\nabla g(2, 1, 3) = \langle 5, -2, 4\rangle$.

   We also know that the plane we are looking for contains the point $(2, 1, 3)$.

   Since we have a normal vector for the plane and a point on the plane, we have enough information to write down an equation of the plane: it is $\boxed{\langle 5, -2, 4\rangle \cdot \langle x - 2, y - 1, z - 3\rangle = 0}$, or $\boxed{5x - 2y + 4z = 20}$.

   (b) *Use linear approximation to approximate $g(2.1, 1.1, 3.1)$.*

   Solution. Linear approximation says that, for (x, y, z) near $(2, 1, 3)$,

$$\begin{aligned} g(x, y, z) &\approx g(2, 1, 3) + g_x(2, 1, 3)(x - 2) + g_y(2, 1, 3)(y - 1) + g_z(2, 1, 3)(z - 3) \\ &= 6 + 5(x - 2) - 2(y - 1) + 4(z - 3) \end{aligned}$$

   So, $g(2.1, 1.1, 3.1) \approx 6 + 5(0.1) - 2(0.1) + 4(0.1) = \boxed{6.7}$.

   Notice that this answer is quite reasonable: $g(2.1, 1.1, 3.1)$ ought to be pretty close to $g(2, 1, 3) = 6$.

   (c) *Suppose the surface in part (a) is the graph of a function $f(x, y)$. Evaluate $D_{\vec{u}} f(2, 1)$, where $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.*

   Solution. We know that $D_{\vec{u}} f(2, 1) = \nabla f(2, 1) \cdot \vec{u}$. So, let's try to find $\nabla f(2, 1)$ using our answer to part (a).

   We are told that $\mathcal{S}$ is the graph of $f(x, y)$, which is the same as saying that $\mathcal{S}$ is described by the equation $z = f(x, y)$ or, equivalently, by the equation $f(x, y) - z = 0$. That is, $\mathcal{S}$ is a level surface of $h(x, y, z) = f(x, y) - z$. Therefore, $\nabla h(2, 1, 3)$ is perpendicular to $\mathcal{S}$ at $(2, 1, 3)$.

   Since $h(x, y, z) = f(x, y) - z$, $\nabla h(2, 1, 3) = \langle f_x(2, 1), f_y(2, 1), -1\rangle$. This vector is normal to the tangent plane $5x - 2y + 4z = 20$ that we found in (a), so it must be parallel to $\langle 5, -2, 4\rangle$. That is, $\langle f_x(2, 1), f_y(2, 1), -1\rangle$ is a scalar multiple of $\langle 5, -2, 4\rangle$. From looking at the z-components of each vector, we can see that this scalar must be $-\frac{1}{4}$. So, $f_x(2, 1) = -\frac{5}{4}$ and $f_y(2, 1) = \frac{1}{2}$, and

$$\begin{aligned} D_{\vec{u}} f(2, 1) &= \nabla f(2, 1) \cdot \vec{u} \\ &= \left\langle -\frac{5}{4}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \boxed{-\frac{7}{20}} \end{aligned}$$

   (d) *Use linear approximation to approximate $f(2.1, 1.1)$.*

Solution. Remember that, if $L(x, y)$ is the linearization of $f(x, y)$ at $(2, 1)$, then $z = L(x, y)$ is the plane tangent to $z = f(x, y)$ at $(2, 1)$. We know that the tangent plane is $5x - 2y + 4z = 20$, which can be rewritten as $z = 5 - \frac{5}{4}x + \frac{1}{2}y$. So, the linearization of $f(x, y)$ at $(2, 1)$ is $5 - \frac{5}{4}x + \frac{1}{2}y$, and the approximate value of $f(2.1, 1.1)$ is $5 - \frac{5}{4}(2.1) + \frac{1}{2}(1.1) = \boxed{2.925}$.[(1)]

Notice that this answer is quite reasonable: $f(2.1, 1.1)$ ought to be quite close to $f(2, 1)$, which is 3 (we know this because $(2, 1, 3)$ is a point on the surface $z = f(x, y)$).

7. *Find the maximum and minimum values of* $f(x, y, z) = x + 2y + z$ *on the surface* $z = 4 - x^2 - y^2$, $z \geq 0$. *(Hint: Parameterize the surface.)*

Solution. Following the hint, let's parameterize the surface. A simple parameterization of the surface is $\vec{r}(u, v) = \langle u, v, 4 - u^2 - v^2 \rangle$, $u^2 + v^2 \leq 4$. The value of f at the point $\vec{r}(u, v)$ is $u + 2v + 4 - u^2 - v^2$, call this $h(u, v)$. We want to maximize and minimize $h(u, v)$ on the domain $u^2 + v^2 \leq 4$.[(2)]

First, we look for critical points (points where $\nabla h = \vec{0}$) in the interior of the domain. Since $\nabla h = \langle 1 - 2u, 2 - 2v \rangle$, the only critical point is $\left(\frac{1}{2}, 1\right)$. This is indeed inside the domain $u^2 + v^2 \leq 4$, so it is a candidate point (a point where the absolute maximum or absolute minimum might be achieved).

Next, we look at the boundary, which is where $u^2 + v^2 = 4$. For this, we use Lagrange multipliers. If we write $g(u, v) = u^2 + v^2$, then we are trying to maximize and minimize $h(u, v)$ subject to the constraint that $g(u, v) = 4$. Therefore, the method of Lagrange multipliers says that we should find points where $\nabla h = \lambda \nabla g$. Since $\nabla g = \langle 2u, 2v \rangle$, we are trying to solve the three equations

$$\begin{cases} 1 - 2u = 2\lambda u \\ 2 - 2v = 2\lambda v \\ u^2 + v^2 = 4 \end{cases}$$

From the first equation, $1 = (2 + 2\lambda)u$, so $u = \frac{1}{2+2\lambda}$.[(3)] Similarly, from the second equation, $v = \frac{2}{2+2\lambda}$. Therefore, $v = 2u$.

Plugging this into the third equation, $5u^2 = 4$, so $u = \pm\frac{2}{\sqrt{5}}$. Combining this with the fact that $v = 2u$, we get two more candidate points: $\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ and $\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$.

Now, we evaluate h on each of our candidate points:

- $h\left(\dfrac{1}{2}, 1\right) = \dfrac{21}{4}$.
- $h\left(\dfrac{2}{\sqrt{5}}, \dfrac{4}{\sqrt{5}}\right) = 2\sqrt{5}$.
- $h\left(-\dfrac{2}{\sqrt{5}}, -\dfrac{4}{\sqrt{5}}\right) = -2\sqrt{5}$.

Therefore, the $\boxed{\text{maximum value is } \frac{21}{4}}$ and the $\boxed{\text{minimum value is } -2\sqrt{5}}$.

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(1) Alternatively, you can use the linear approximation formula $f(x, y) \approx f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1)$ to approximate $f(2.1, 1.1)$.

(2) Since this domain is closed and bounded, the Extreme Value Theorem for Functions of Two Variables says that h will attain an absolute maximum value and an absolute minimum value on the domain.

(3) We don't need to worry about dividing by 0 here: since the product of $2 + 2\lambda$ and u is non-zero, both $2 + 2\lambda$ and u must be non-zero.