

- 1 (a) Find the curvature $\kappa(t)$ of the curve $\mathbf{r}(t) = \langle -\cos t, \sin t, -2t \rangle$ at the point corresponding to $t = 0$.

Hint: You may use the two formulas for the curvature $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

Solution: We compute:

$$\begin{aligned}\mathbf{r}'(t) &= \langle \sin t, \cos t, -2 \rangle \\ \mathbf{r}''(t) &= \langle \cos t, -\sin t, 0 \rangle.\end{aligned}$$

Therefore

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & -2 \\ \cos t & -\sin t & 0 \end{vmatrix} = \langle -2 \sin t, -2 \cos t, -1 \rangle.$$

and so

$$\begin{aligned}\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\ &= \frac{|\langle -2 \sin t, -2 \cos t, -1 \rangle|}{|\langle \sin t, \cos t, -2 \rangle|^3} \\ &= \frac{\sqrt{5}}{(\sqrt{5})^3} = \frac{1}{5}.\end{aligned}$$

- (b) Find the osculating plane of this curve at the point corresponding to $t = 0$.

Solution: The osculating plane is perpendicular to $\mathbf{T}(t)$ and $\mathbf{N}(t)$, but it is also perpendicular to $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$, and those are easier to work with. We've already computed a normal vector $\mathbf{n} = \mathbf{r}'(0) \times \mathbf{r}''(0)$, which is $\langle -2 \sin(0), -2 \cos(0), -1 \rangle = \langle 0, -2, -1 \rangle$. This plane passes through the point $\mathbf{r}(0) = \langle -\cos 0, \sin 0, -2(0) \rangle = \langle -1, 0, 0 \rangle$, so the plane has equation

$$\langle 0, -2, -1 \rangle \cdot \langle x - (-1), y, z \rangle = 0 \quad \text{or} \quad -2y - z = 0 \quad \text{or} \quad 2y + z = 0.$$

- (c) Find the normal plane of this curve at the point corresponding to $t = 0$.

Solution: The normal plane is perpendicular to $\mathbf{T}(0)$, which points in the same direction as $\mathbf{r}'(0)$. Thus the normal plane is through the point $\mathbf{r}(0) = \langle -1, 0, 0 \rangle$ and perpendicular to $\mathbf{r}'(0) = \langle \sin(0), \cos(0), -2 \rangle = \langle 0, 1, -2 \rangle$. Thus the plane has equation

$$\langle 0, 1, -2 \rangle \cdot \langle x - (-1), y, z \rangle = 0 \quad \text{or} \quad y - 2z = 0.$$

- (d) Find the curvature $\kappa(t)$ of the curve $\mathbf{r}(t) = \langle -\cos(5t), \sin(5t), -10t \rangle$ at the point corresponding to $t = 0$.

Solution: This is a reparameterization of the curve in part (a) (we've replaced t with $5t$). Thus the curve traced out by this vector function is the same and hence it has the same curvature: $\kappa(t) = \frac{1}{5}$.

- 2 Consider the plane $x + 3z = 1$ and the line parameterized by $x = 2t + 1$, $y = 4t - 1$, $z = 3t$.

- (a) Find two different unit vectors that are normal to the plane.

Solution: There are only two: $\frac{1}{\sqrt{10}} \langle 1, 0, 3 \rangle$ and $-\frac{1}{\sqrt{10}} \langle 1, 0, 3 \rangle$.

- (b) Find a vector $\mathbf{a}$ that is parallel to the line.

Solution: The line is $\langle x, y, z \rangle = \langle 2t+1, 4t-1, 3t \rangle$ or $\mathbf{r}(t) = \langle 1, -1, 0 \rangle + t\langle 2, 4, 3 \rangle$. Thus a vector parallel to the line is $\mathbf{a} = \langle 2, 4, 3 \rangle$ (or any scalar multiple of this).

- (c) Explain why the line is not parallel to the plane.

Solution: The easiest way to see this is to see that $\mathbf{a} \cdot \mathbf{n} \neq 0$, where $\mathbf{n}$ is a normal to the plane. We'll use $\mathbf{n} = \langle 1, 0, 3 \rangle$ for simplicity; we get $\mathbf{a} \cdot \mathbf{n} = \langle 2, 4, 3 \rangle \cdot \langle 1, 0, 3 \rangle = 11$, not zero.

- (d) Find the point of intersection of the line and the plane.

Solution: One way to do this is to plug in $x = 2t + 1$, $y = 4t - 1$, $z = 3t$ into the equation for the plane and then find t . We find $t = 0$, so $(x, y, z) = (1, -1, 0)$ is the point of intersection.

- (e) Find an equation for a plane that contains both the line and the origin.

Solution: The line contains the vector $\mathbf{a} = \langle 2, 4, 3 \rangle$ and the point $(x, y, z) = (1, -1, 0)$. Thus the plane must contain the vector $\langle 1, -1, 0 \rangle$ from the origin to our point. We can then compute our plane's normal vector:

$$\mathbf{n} = \langle 2, 4, 3 \rangle \times \langle 1, -1, 0 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ 1 & -1 & 0 \end{vmatrix} = \langle 3, 3, -6 \rangle.$$

Since the plane contains the origin, we can write the equation as

$$\langle 3, 3, -6 \rangle \cdot \langle x, y, z \rangle = 0 \quad \text{or} \quad x + y - 2z = 0.$$

- (f) Find the angle between the plane in part (e) and the original plane, $x + 3z = 1$. (Recall that the angle between two planes is defined as the acute angle between their normal vectors.) If you weren't able to find the plane in part (e), then use the plane $-2x + 7y - 12z = 0$ instead.

Solution: We can find the angle θ between the two normal vectors using either the dot product (which will give us $\cos \theta$) or the cross product ($\sin \theta$). We'll use the dot product since it's easier to compute:

$$\cos(\theta) = \frac{\langle 3, 3, -6 \rangle \cdot \langle 1, 0, 3 \rangle}{|\langle 3, 3, -6 \rangle| |\langle 1, 0, 3 \rangle|} = \frac{-15}{(3\sqrt{6})(\sqrt{10})} = -\frac{\sqrt{5}}{2\sqrt{3}}.$$

Before we get too excited, we should note that the answer is **NOT** simply $\cos^{-1}\left(-\frac{\sqrt{5}}{2\sqrt{3}}\right)$, since this is not an acute angle. Instead our answer is that the angle between the planes is $\pi - \cos^{-1}\left(-\frac{\sqrt{5}}{2\sqrt{3}}\right)$.

3 Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors such that $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

- (a) What is $(-3\mathbf{b}) \times (4\mathbf{a})$?

Solution: $(-3\mathbf{b}) \times (4\mathbf{a}) = -12\mathbf{b} \times \mathbf{a} = 12\mathbf{a} \times \mathbf{b} = 12\langle 2, 2, -3 \rangle = \langle 24, 24, -36 \rangle$.

- (b) Find $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ if it exists. If it doesn't exist then explain why not.

Solution: This is zero. $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to $\mathbf{a}$ (and $\mathbf{b}$, not that it matters), so $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

- (c) Find $\mathbf{a} \times (\mathbf{a} \cdot \mathbf{b})$ if it exists. If it doesn't exist then explain why not.

Solution: This doesn't exist: $\mathbf{a} \cdot \mathbf{b}$ is a scalar (a number), and we can't cross a vector with a scalar.

- (d) Suppose that not only does $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, but also that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}|$. Which of the following statements *must* be true? (Simply circle all the statements that must be true – no explanations are necessary.)

(i) $\mathbf{a} \neq \mathbf{b}$

(ii) $\mathbf{a}$ and $\mathbf{b}$ are parallel

(iii) $\mathbf{a}$ is perpendicular to $\mathbf{b}$

(iv) either $\mathbf{a}$ or $\mathbf{b}$ must be a unit vector

Solution: Recall that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin(\theta)$, so for $\mathbf{a}$ and $\mathbf{b}$ we must have $\sin \theta = 1$. Thus we have two true statements:

(i) $\mathbf{a} \neq \mathbf{b}$

(iii) $\mathbf{a}$ is perpendicular to $\mathbf{b}$

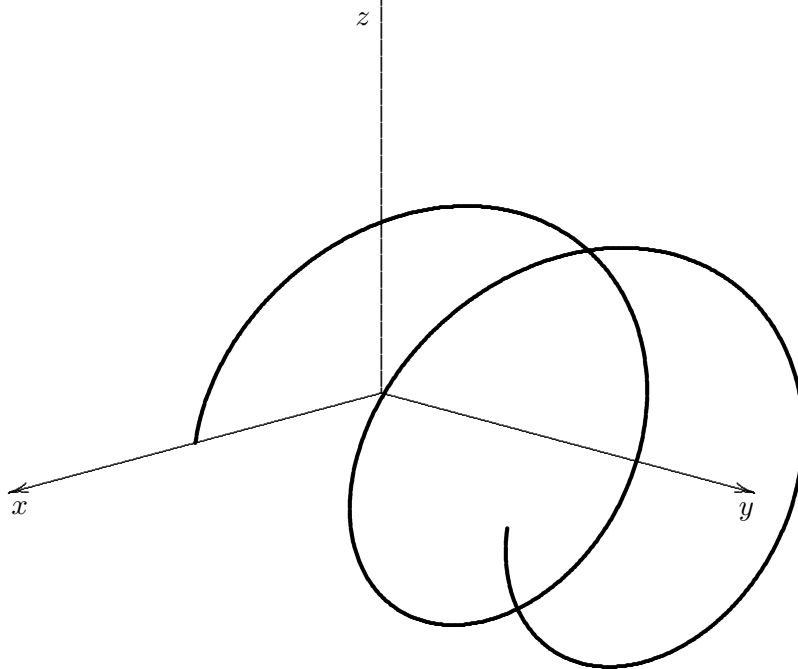
- 4 Consider the space curve defined by the vector function

$$\mathbf{r}(t) = \langle 3 \cos(2t), 2t, 3 \sin(2t) \rangle$$

with $0 \leq t \leq 2\pi$.

- (a) Sketch the curve as neatly as possible. (Be sure to label your axes.)

Solution: This is a helix, starting at $(3, 0, 0)$ and winding its way around the positive y -axis to the point $(3, 4\pi, 0)$:



- (b) Find an equation for a surface that this space curve lies on and describe the surface.

Solution: Since $(3 \cos(2t))^2 + (3 \sin(2t))^2 = 9$, we get $x^2 + z^2 = 9$. This is a cylinder centered on the y -axis with radius 3.

- (c) Find the length of this space curve.

Solution: The length of a smooth curve from $t = a$ to $t = b$ is $\int_a^b |\mathbf{r}'(t)| dt$. Here $\mathbf{r}'(t) = \langle -6 \sin(2t), 2, 6 \cos(2t) \rangle$, so $|\mathbf{r}'(t)| = \sqrt{40} = 2\sqrt{10}$. Thus the length of the curve is

$$\int_0^{2\pi} 2\sqrt{10} dt = 4\pi\sqrt{10}.$$

- (d) Give parametric equations for the tangent line to this space curve at the point $(3, 2\pi, 0)$.

Solution: The tangent line requested goes through the point $(3, 2\pi, 0)$ when $t = \pi$, so it has direction equal to $\mathbf{r}'(\pi) = \langle -6 \sin(2\pi), 2, 6 \cos(2\pi) \rangle = \langle 0, 2, 6 \rangle$. Thus one set of parametric equations for the tangent line is $\mathbf{x}(t) = \langle 3, 2\pi, 0 \rangle + t\langle 0, 2, 6 \rangle$.

- 5 (a) Consider the space curve that is defined as the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$. Explain why this space curve can be parameterized by the vector function $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ with $0 \leq t \leq 2\pi$.

Solution: This space curve, when projected vertically into the xy -plane, is simply the unit circle $x^2 + y^2 = 1$, which we parameterize $x = \cos t$ and $y = \sin t$. The z component can be found because the curve lies in the plane $z = y$, so $z = \sin t$.

- (b) Now consider the vector function $\mathbf{p}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$, where $\mathbf{r}(t)$ is the vector function given in part (a). Compute the components of the vector function $\mathbf{p}(t)$.

Solution: Since $|\mathbf{r}(t)| = \sqrt{\cos^2 t + 2 \sin^2 t}$, the vector function $\mathbf{p}(t)$ is

$$\mathbf{p}(t) = \left\langle \frac{\cos t}{\sqrt{\cos^2 t + 2 \sin^2 t}}, \frac{\sin t}{\sqrt{\cos^2 t + 2 \sin^2 t}}, \frac{\sin t}{\sqrt{\cos^2 t + 2 \sin^2 t}} \right\rangle.$$

- (c) Now consider the space curve formed by the intersection of the same plane $y = z$ and the unit sphere (the sphere of radius one centered at the origin). Explain why this new space curve is parameterized by the vector function $\mathbf{p}(t)$.

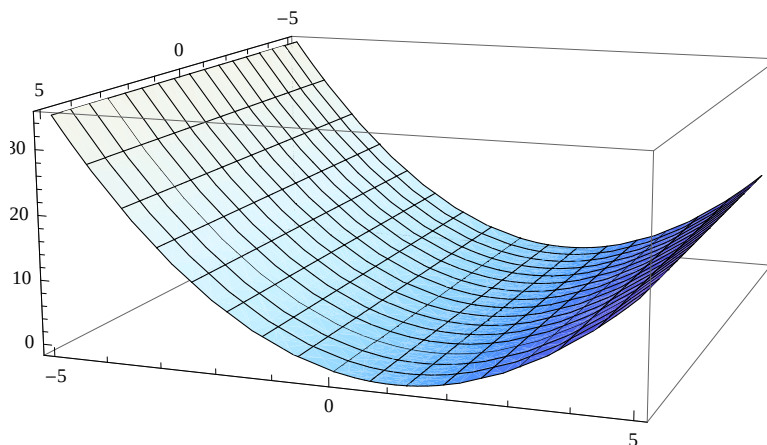
Solution: We only need to confirm two facts:

- The space curve $\mathbf{p}(t)$ lies on the plane $y = z$, and
- The space curve $\mathbf{p}(t)$ lies on the unit sphere.

The first is obvious (look at the y and z components of $\mathbf{p}(t)$). The second isn't very hard: $|\mathbf{p}(t)| = \frac{|\mathbf{r}(t)|}{|\mathbf{r}(t)|} = 1$, so the space curve lies on this sphere.

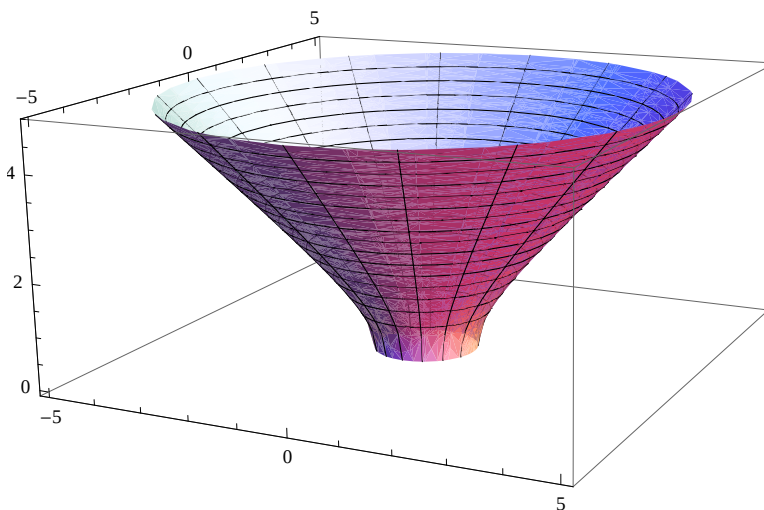
- 6 (a) Sketch the graph of the function $f(x, y) = (y - 1)^2 - 1$. Make your sketch as neat as possible, and be sure to label your axes carefully.

Solution: In the yz -plane, this is simply a parabola. In fact, the $x = k$ is always the same parabola. Thus this is a parabolic “cylinder” as seen here:



- (b) Now sketch the graph of the function $f(x, y) = \sqrt{x^2 + y^2 - 1}$. Again, make your sketch as neat as possible, and label your axes carefully.

Solution: If we write $z = \sqrt{x^2 + y^2 - 1}$, we can re-arrange this to be $x^2 + y^2 - z^2 = 1$, a one-sheeted hyperboloid. But since the square root function outputs non-negative numbers, we only get half this hyperboloid:



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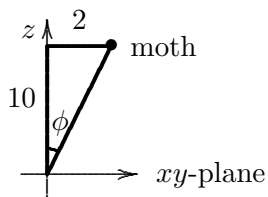
A moth is flying around a light bulb hanging from the ceiling of a room. The light bulb is 10 feet above the floor and the moth stays exactly 10 feet above the floor as it flies around the light bulb. The moth flies in a circular path, at constant speed, always exactly 2 feet away from the center of the light bulb.

- (a) Write down a set of equations that describe the circle traced out by the moth in spherical coordinates. Consider the origin to be the point on the floor directly below the light bulb.

Solution: The circle in spherical coordinates is simply $\rho = \sqrt{104} = 2\sqrt{26}$ and $\phi = \tan^{-1}(\frac{1}{5})$. (The angle θ is arbitrary.) If it helps, we can write this in rectangular coordinates:

$$\begin{aligned} \langle x, y, z \rangle &= \left\langle 2\sqrt{26} \sin\left(\tan^{-1}\left(\frac{1}{5}\right)\right) \cos(\theta), 2\sqrt{26} \sin\left(\tan^{-1}\left(\frac{1}{5}\right)\right) \sin(\theta), 2\sqrt{26} \cos\left(\tan^{-1}\left(\frac{1}{5}\right)\right) \right\rangle \\ &= \langle 2 \cos(\theta), 2 \sin(\theta), 10 \rangle. \end{aligned}$$

Here's a small picture of the situation:



The hypotenuse of this right triangle is $\rho = \sqrt{104} = 2\sqrt{26}$ and $\tan(\phi) = \frac{2}{10} = \frac{1}{5}$. These values are what produced the spherical coordinates above.

- (b) Write down the equations giving the moth's circular path in cylindrical coordinates, and then give a parameterization of the moth's path in cylindrical coordinates.

Solution: In cylindrical coordinates this path is $r = 2$, $z = 10$. A parameterization in cylindrical coordinates might be $(r, \theta, z) = (2, t, 10)$ (simply $\theta = t$ together with the equations involving r and z , above).

(c) Suppose that the moth's flight path is given by a vector function $\mathbf{r}(t)$, with parameter t representing time. Given what you know about the moth's flight answer the following questions:

(i) Is it possible to determine $\mathbf{r}'(t) \cdot \mathbf{k}$? If so, what is it?

Solution: Yes, $\mathbf{r}'(t) \cdot \mathbf{k} = 0$. The moth always flies at the same height, so the tangent vector (velocity) has is horizontal (has no vertical component). Thus this velocity is perpendicular to the vertical vector $\mathbf{k}$, so the dot product is zero.

(ii) Describe the curve parameterized by $\mathbf{r}'(t) \times \mathbf{k}$.

Solution: We parameterize the curve traced by the moth by

$$\mathbf{r}(t) = \langle 10 \cos(t), 10 \sin(t), 2 \rangle.$$

(Note we're assuming that the angular speed is 1, but we have no units, so why not?) Then $\mathbf{r}'(t) = \langle -10 \sin(t), 10 \cos(t), 0 \rangle$ and so

$$\mathbf{r}'(t) \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 \sin(t) & 10 \cos(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 10 \cos(t), 10 \sin(t), 0 \rangle.$$

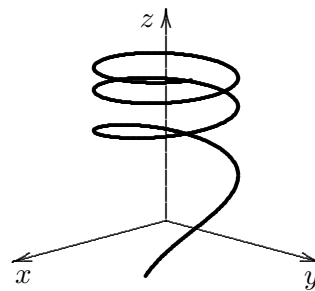
This curve is a circle in the xy -plane.

8 Match the following space curves to the appropriate vector functions. (Some of the graphs appear on the next page.)

Solution:

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, \ln t \rangle$

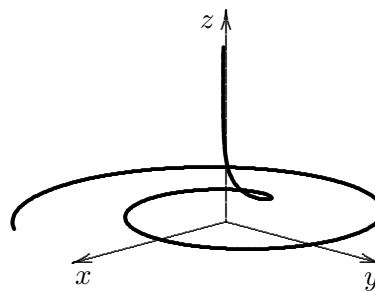
This is Curve III. The distinguishing characteristic is that, as the x and y coordinates revolve in a circle, the z coordinate increases quickly from $-\infty$ then increases slower and slower.



Curve III

(b) $\mathbf{r}(t) = \langle t \cos t, t \sin t, \frac{1}{t} \rangle$

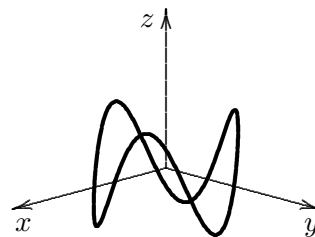
This is Curve I. The key thing to note is that when x and y are at radius t , the height is $z = \frac{1}{t}$. So as t decreases to zero, x and y like on a very small circle as z grows without bound.



Curve I

(c) $\mathbf{r}(t) = \langle \cos t, \sin t, \sin(3t) \rangle$

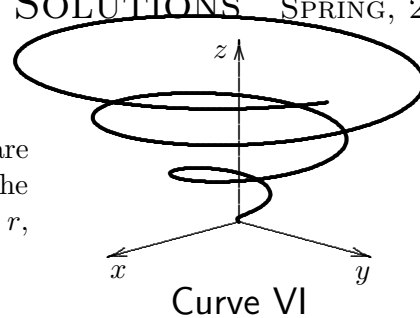
This is Curve II. As x and y travel around their customary circle, the z coordinate rises and falls through three cycles.



Curve II

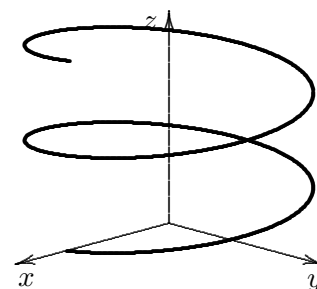
(d) $\mathbf{r}(t) = \langle t \cos t, t \sin t, t \rangle$

The key feature to notice is that the x and y coordinates are revolving with an ever-increasing radius that happens to be the same as the z coordinate. Thus the curve lies on the cone $z = r$, and this is Curve VI.



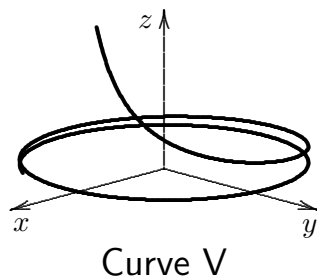
(e) $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

This is our old friend the helix, Curve IV.



(f) $\mathbf{r}(t) = \langle \cos t, \sin t, \frac{1}{t} \rangle$

This is Curve V, the only one left. The key here is that the height $z = \frac{1}{t}$, the inverse of the angle through which the curve has revolved around the z -axis. Thus as t decreases to zero, for example, z grows without bound. As t grows larger, however, the height z decreases toward zero.



- 9 A planar mirror in space contains the point $P(4, 1, 5)$ and is perpendicular to the vector $\mathbf{n} = \langle 1, 2, -3 \rangle$. The light ray $\overrightarrow{QP} = \mathbf{v} = \langle -3, 1, -2 \rangle$ with source $Q(7, 0, 7)$ hits the mirror plane at the point P .

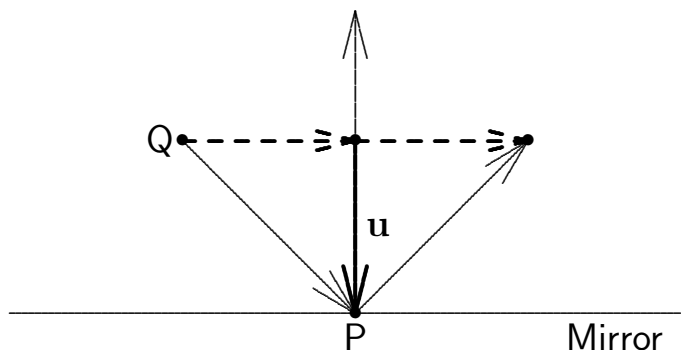
- (a) Compute the projection $\mathbf{u} = \text{proj}_{\mathbf{n}}(\mathbf{v})$ of $\mathbf{v}$ onto $\mathbf{n}$.

Solution: Recall that $\text{proj}_{\mathbf{n}}(\mathbf{v}) = \frac{\mathbf{n} \cdot \mathbf{v}}{|\mathbf{n}|^2} \mathbf{n}$. Since $\mathbf{n} = \langle 1, 2, -3 \rangle$ and $\mathbf{v} = \langle -3, 1, -2 \rangle$, we get

$$\mathbf{u} = \text{proj}_{\mathbf{n}}(\mathbf{v}) = \frac{\langle 1, 2, -3 \rangle \cdot \langle -3, 1, -2 \rangle}{|\langle 1, 2, -3 \rangle|^2} \langle 1, 2, -3 \rangle = \frac{5}{14} \langle 1, 2, -3 \rangle.$$

- (b) Identify $\mathbf{u}$ in the figure below and use it to find a vector parallel to the reflected ray.

Solution: I've re-drawn the figure faintly and drawn in $\mathbf{u}$ as more dark.



The two dashed arrows are both $\overrightarrow{QP} - \mathbf{u}$, so the vector we want is $(-\mathbf{u}) + (\overrightarrow{QP} - \mathbf{u}) = \overrightarrow{QP} - 2\mathbf{u}$.

- 10 What is the distance between the two cylinders $x^2 + y^2 = 1$ and $(z - 2)^2 + (x - 5)^2 = 4$?

Solution: The first cylinder is centered along the z -axis and has radius 1. The second is centered parallel to the y -axis, but with $x = 5$ and $z = 2$; it has radius 2. We'll find the distance between these two center lines and subtract 3 ($= 1 + 2$).

The distance between the lines is 5, and so the distance between the cylinders is 2. (One can compute the distance between the lines in the usual way, but it's easier just to think about it. When are the two lines closest?)

- 11 (a) Parameterize the intersection of the ellipsoid

$$\frac{x^2}{4} + \frac{(y - 5)^2}{4} + \frac{z^2}{9} = 2$$

with the plane $z = 3$.

Solution: When $z = 3$, the x and y coordinates satisfy

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y - 5}{2}\right)^2 = 1.$$

We parameterize this circle with $\frac{x}{2} = \cos t$ and $\frac{y - 5}{2} = \sin t$, so the full parameterization is

$$\mathbf{r}(t) = \langle 2 \cos t, 5 + 2 \sin t, 3 \rangle.$$

- (b) Parameterize the ellipsoid itself in the form $\mathbf{r}(\theta, \phi) = \dots$.

Solution: In order to use spherical coordinates, we make the equation look like a sphere:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y - 5}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = (\sqrt{2})^2.$$

Using $\rho = \sqrt{2}$, we get

$$\begin{aligned} \frac{x}{2} &= \sqrt{2} \sin(\phi) \cos(\theta) \\ \frac{y - 5}{2} &= \sqrt{2} \sin(\phi) \sin(\theta) \\ \frac{z}{3} &= \sqrt{2} \cos(\phi), \end{aligned}$$

or

$$\mathbf{r}(\theta, \phi) = \left\langle 2\sqrt{2} \sin(\phi) \cos(\theta), 5 + 2\sqrt{2} \sin(\phi) \sin(\theta), 3\sqrt{2} \cos(\phi) \right\rangle.$$

- (c) What is the curvature of the curve in part (a) at the point $(2, 0, 3)$.

Hint: While you may use the curvature formula $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$, you are also allowed to cite a fact which you know about the curvature.

Solution: Rather than just plunging into the computation, we'll instead infer from the hint that there's a simpler way to do this. The curve in question is a circle of radius 2 and we recall that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$. Thus the curvature of our circle is $\frac{1}{2}$ for all values of t .