MATH 21A   OLD EXAM ONE – SPRING 2007 SOLUTIONS SPRING, 2009

PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

1 The vectors $\mathbf{A} = -2\mathbf{i} + (t-1)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = \mathbf{j} + t\mathbf{k}$ are parallel when:

(a) $t = 0$;

(b) $t = \frac{1}{3}$;

(c) $t = 1$ and $t = 2$;

(d) all values of t;

(e) no values of t.

Solution: The answer is (e). The vectors are never parallel because $\mathbf{A}$ always has a non-zero $\mathbf{i}$ component whereas $\mathbf{B}$ always has a zero $\mathbf{i}$ component.

2 If $\mathbf{A}$ and $\mathbf{B}$ are vectors in space, then the expression $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$ equals

(a) $\mathbf{0}$ (the zero vector) because $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$ are always parallel;

(b) 0 (the number zero) because $\mathbf{A}$ and $\mathbf{B}$ are always perpendicular to $\mathbf{A} \times \mathbf{B}$;

(c) a positive number since $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$ is the volume of a parallelepiped;

(d) undefined since we cannot take the dot product of a vector and a scalar;

(e) an arbitrary number, depending on the particular vectors $\mathbf{A}$ and $\mathbf{B}$.

Solution: The answer is (b). The vector $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane parallel to $\mathbf{A}$ and $\mathbf{B}$. The sum $\mathbf{A} + \mathbf{B}$ lies parallel to this plane as well, so $\mathbf{A} + \mathbf{B}$ is perpendicular to $\mathbf{A} \times \mathbf{B}$.

3 The curve $\mathbf{r}(t) = \langle e^{\sqrt{2}\,t}\cos t, e^{\sqrt{2}\,t}\sin t\rangle$, $0 \le t \le \pi$, has length

(a) $\frac{1}{\sqrt{3}}\left(e^{\sqrt{2}\,\pi} - 1\right)$;

(b) $\frac{1}{\sqrt{2}}\left(e^{\sqrt{2}\,\pi} - 1\right)$;

(c) $\sqrt{\frac{2}{3}}\left(e^{\sqrt{2}\,\pi} - 1\right)$;

(d) $\sqrt{\frac{3}{2}}\left(e^{\sqrt{2}\,\pi} - 1\right)$;

(e) given by a definite integral which cannot be evaluated explicitly;

(f) given by an indefinite integral.

Solution: The arc length is given by the definite integral $\int_0^\pi |\mathbf{r}'(t)|\,dt$ (which is often difficult to compute). In this case,

$$\mathbf{r}'(t) = \langle \sqrt{2}e^{\sqrt{2}\,t}\cos t - e^{\sqrt{2}\,t}\sin t, \sqrt{2}e^{\sqrt{2}\,t}\sin t + e^{\sqrt{2}\,t}\cos t\rangle$$
$$= e^{\sqrt{2}\,t}\langle \sqrt{2}\cos t - \sin t, \sqrt{2}\sin t + \cos t\rangle.$$

Therefore

$$|\mathbf{r}'(t)|^2 = e^{2\sqrt{2}\,t}\left[\left(\sqrt{2}\cos t - \sin t\right)^2 + \left(\sqrt{2}\sin t + \cos t\right)^2\right]$$
$$= e^{2\sqrt{2}\,t}\left[\left(2\cos^2 t - 2\sqrt{2}\sin t\cos t + \sin^2 t\right) + \left(2\sin^2 t + 2\sqrt{2}\sin t\cos t + \cos^2 t\right)\right]$$
$$= 3e^{2\sqrt{2}\,t}$$

Thus $|\mathbf{r}'(t)| = \sqrt{3}e^{\sqrt{2}\,t}$ and the arc length is

$$\int_0^\pi \sqrt{3}e^{\sqrt{2}\,t}\,dt = \sqrt{\frac{3}{2}}e^{\sqrt{2}\,t}\Big|_0^\pi = \sqrt{\frac{3}{2}}\left(e^{\sqrt{2}\,\pi} - 1\right),$$

and our answer is (d).

The questions in problems 4-5 refer to the marked points in the diagram below, which shows the level curves of a function $f(x,y)$.

4 At which of the labeled points is the partial derivative $\frac{\partial f}{\partial x}$ strictly positive?

(a) at S and T (b) at S and W

(c) at T and W (d) at V and W

(e) at V and Y (f) only at V

(g) at W and Y (h) only at T

Solution: The correct answer is (e) at V and Y.

5 At the point Y, what can one say about the values of $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y^2}$?

(a) $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial^2 f}{\partial y^2} > 0$ (b) $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial^2 f}{\partial y^2} < 0$

(c) $\frac{\partial f}{\partial y} < 0$ and $\frac{\partial^2 f}{\partial y^2} > 0$ (d) $\frac{\partial f}{\partial y} < 0$ and $\frac{\partial^2 f}{\partial y^2} < 0$

(e) $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial^2 f}{\partial y^2} > 0$ (f) $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial^2 f}{\partial y^2} < 0$

(g) $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial^2 f}{\partial y^2} = 0$ (h) $\frac{\partial f}{\partial y} < 0$ and $\frac{\partial^2 f}{\partial y^2} = 0$

Solution: The correct answer is (b): f_Y is positive but f_{yy} is negative.

6 A particle moves in space with position vector $\mathbf{r}(t)$. If $\mathbf{r}''(t) \perp \mathbf{r}'(t)$ for all t, what can you conclude about the motion?
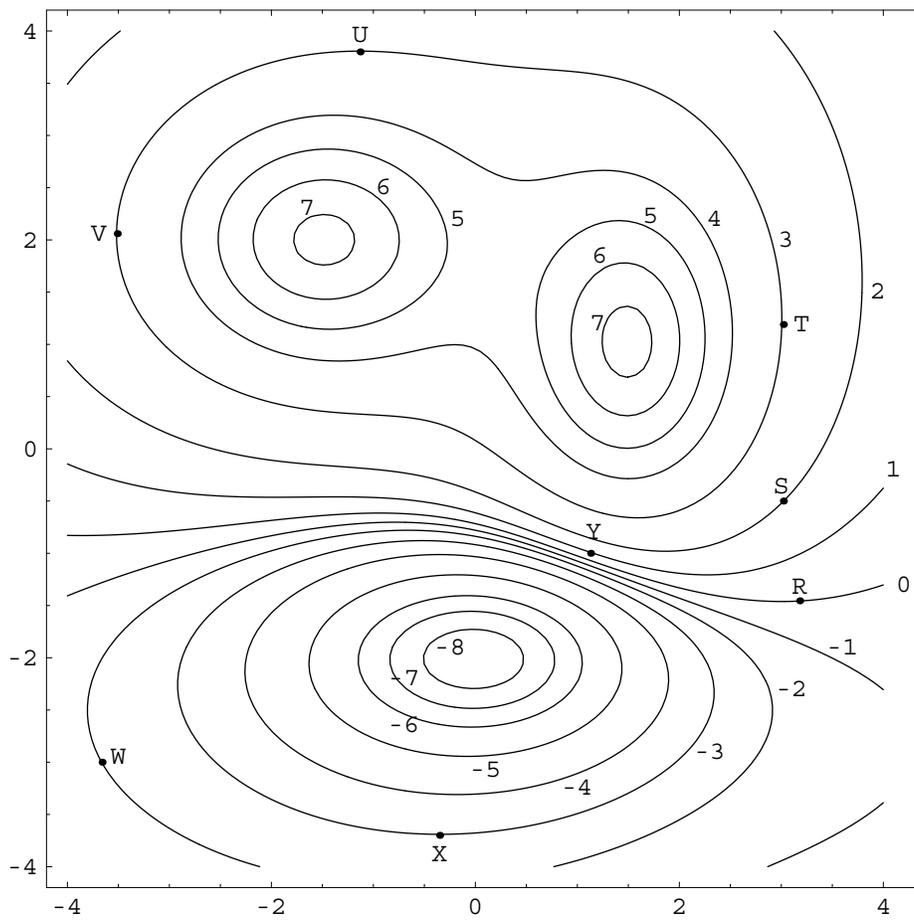
(a) $\mathbf{r}(t) \times \mathbf{r}'(t)$ is constant;

(b) $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is constant;

(c) $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant;

(d) $|\mathbf{r}'(t)|^2 = \mathbf{r}'(t) \cdot \mathbf{r}'(t)$ is constant;

(e) $\mathbf{r}'(t)$ is constant;

(f) $\mathbf{r}''(t)$ is constant;

Solution: This is a little tricky, but you might recall this particular trick from the proof that $\mathbf{T}'$ is perpendicular to $\mathbf{T}$.

That $\mathbf{r}''(t) \perp \mathbf{r}'(t)$ for all t means that $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$ always. But

$$\frac{d}{dt}\left(|\mathbf{r}'(t)|^2\right) = \frac{d}{dt}\left(\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right) = \mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0.$$

Thus $|\mathbf{r}'(t)|$ is constant, so (d) is our answer.

PART II: Free response questions. You should attempt all parts of each problem. Show your work!

7 Let Π_1 be the plane described by the equation $x - y + z = 5$ and Π_2 the plane passing through the three points $(2, 1, 0)$, $(1, 1, 1)$, $(0, 0, 1)$.

(a) Find an equation describing the plane Π_2.

Solution: The plane Π_2 contains the vectors $\langle 1, 0, -1 \rangle$ and $\langle 1, 1, 0 \rangle$, so it has normal vector

$$\mathbf{n} = \langle 1, 0, -1 \rangle \times \langle 1, 1, 0 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \langle 1, -1, 1 \rangle.$$

Thus various equations for this plane are

$$\langle 1, -1, 1 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0 \qquad \text{or} \qquad x - y + z = 1.$$

(b) Show that the planes Π_1 and Π_2 are parallel.

Solution: Both planes are perpendicular to $\langle 1, -1, 1 \rangle$, so the two planes are parallel.

(c) Find the distance between Π_1 and Π_2. (Hint: The distance between two parallel planes equals the distance between any particular point on one plane to the other plane.)

Solution: We'll follow the hint and simply find the distance between the point $(0, 0, 1)$ and the plane Π_1. Label this point P, and recall that the normal vector for Π_1 is $\mathbf{n} = \langle 1, -1, 1 \rangle$. Let Q be *any* point on Π_1 – we'll take $(5, 0, 0)$ for Q; then the distance from P to the plane is simply the (absolute value of the) scalar projection $|\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP}| = |\overrightarrow{QP}||\cos(\theta)| = \left| \frac{\mathbf{n} \cdot \overrightarrow{QP}}{|\mathbf{n}|} \right|$:
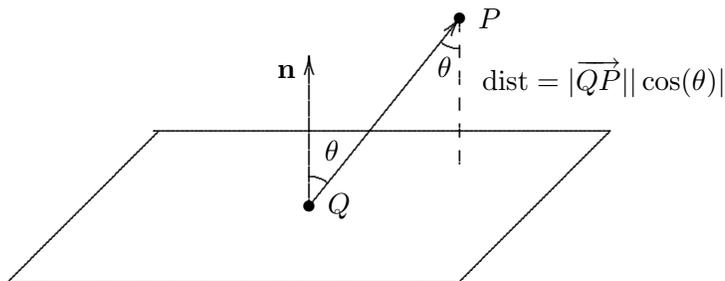
In this case, $\overrightarrow{QP} = \langle -5, 0, 1 \rangle$ and $\mathbf{n} = \langle 1, -1, 1 \rangle$, so $\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP} = \frac{\mathbf{n} \cdot \overrightarrow{QP}}{|\mathbf{n}|} = \frac{-4}{\sqrt{3}}$. Thus the distance is $\frac{4}{\sqrt{3}}$.

8 An airplane traces out the helix $x = \cos t$, $y = \sin t$, $z = 2\pi - t$, $(0 \le t \le 2\pi)$. As it travels, it emits a laser ray shining straight ahead – that is, shining in the direction of the velocity vector.

(a) Find the velocity vector of the airplane at time t.

Solution: This is $\mathbf{r}'(t) = \langle -\sin t, \cos t, -1 \rangle$, where $\mathbf{r}(t) = \langle x, y, z \rangle = \langle \cos t, \sin t, 2\pi - t \rangle$.

(b) Give a linear parameterization of the laser ray emitted at time $t = \pi$. Where does this ray intersect the xy-plane?

Solution: The line passes through the point $\mathbf{r}(\pi) = \langle -1, 0, \pi \rangle$ and is in the direction of $\mathbf{r}'(\pi) = \langle 0, -1, -1 \rangle$. Thus the line is

$$\mathbf{x}(u) = \langle -1, 0, \pi \rangle + u\langle 0, -1, -1 \rangle.$$

(c) Give a linear parameterization of the laser ray emitted at time t. (Now t is not a specific number!) Hint: use a different letter to denote the parameter for the tangent line.

Solution: This is

$$\mathbf{x}(u) = \mathbf{r}(t) + u\mathbf{r}'(t)$$
$$= \langle \cos t, \sin t, 2\pi - t \rangle + u\langle -\sin t, \cos t, -1 \rangle.$$

(d) As the airplane travels along the helix, the spots where the laser rays intersect the xy-plane trace out a curve in the xy-plane. Give a parameterization of this plane curve.

Solution: The curve $\mathbf{x}(u)$ from part (c) hits the xy-plane when the z-coordinate $2\pi - t - u = 0$, when $u = 2\pi - t$. Thus the curve in the xy-plane is parameterized by

$$\langle \cos t, \sin t, 2\pi - t \rangle + (2\pi - t)\langle -\sin t, \cos t, -1 \rangle.$$

9 (a) In *only a few words*, describe the shape of the surface S, given parametrically by the equation

$$\mathbf{r}(\theta, t) = \langle \tfrac{t}{2} \cos \theta, \tfrac{t}{3} \sin \theta, t^2 \rangle, \qquad -\pi \le \theta < \pi, \quad t \ge 0.$$

Solution: This is an elliptical paraboloid. The $t = k$ grid curves are ellipses in the $z = k^2$ plane, whereas the $\theta = k$ grid curves are halves of parabolas in the the $\theta = k$ half-plane.

(b) Verify that the point $P = (\tfrac{1}{2}, \tfrac{1}{3}, 2)$ lies on the surface S.

Solution: We solve for t and θ in the equations

$$\left\langle \frac{t}{2} \cos \theta, \frac{t}{3} \sin \theta, t^2 \right\rangle = \left\langle \frac{1}{2}, \frac{1}{3}, 2 \right\rangle.$$

From the z-coordinate, we get $t = \sqrt{2}$. Thus we get $\cos \theta = \sin \theta = \frac{1}{\sqrt{2}}$, so $\theta = \frac{\pi}{4}$. Thus our point is $\mathbf{r}(\frac{\pi}{4}, \sqrt{2})$.

(c) Give an alternate description of the surface S of part (a) as the graph of a function of x and y. That is, find an equation $z = f(x, y)$ in terms of the Euclidean coordinates x, y, z, but not involving the parameters θ, t. Hint: you may find the relation $\cos^2 \theta + \sin^2 \theta = 1$ useful.

Solution: Notice that $(2x)^2 + (3y)^2 = t^2$ and $z = t^2$, so we get the equation $z = 4x^2 + 9y^2$.

(d) Find a linear equation for the tangent plane to S at the point P of part (b). You can do this using either the parameterization of S in part (a), or your answer to part (c).

Solution: This is from material we'll cover after the midterm. Sorry about that!

10 Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

Solution: A vector $\mathbf{v}$ parallel to the line of intersection must be perpendicular to the normal vectors of both planes. Thus we can use the cross product to find this vector:

$$\mathbf{v} = \langle 2, 1, 1 \rangle \times \langle 1, 3, 1 \rangle \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \langle -2, -1, 5 \rangle.$$

This is the normal vector for the requested plane, so our plane is $-2x - y + 5z = 0$.

11 For each of the following, fill in the blank with $<$ (less than), $>$ (greater than), or $=$ (equals). Justify your answer completely.

(a) The arc length of the curve parameterized by $\mathbf{f}(t) = \langle\cos(2t), 0, \sin(2t)\rangle$, $0 \le t \le \pi$.

$=$

The arc length of the curve parameterized by $\mathbf{g}(u) = \langle 3, 2\cos(u^2), 2\sin(u^2)\rangle$, $0 \le u \le \sqrt{\pi}$.

Comments: Both these arcs are parts of circles – the one on the left is a full circle of radius 1 and the one on the right is a half circle of radius 2. Thus both have arc length 2π.

(b) The arc length of the curve parameterized by $\mathbf{f}(t) = \langle t^2, 2\cos(t), 2\sin(t)\rangle$, $0 \le t \le 2\pi$.

$<$

The   arc   length   of   the   curve parameterized   by $\mathbf{g}(u)$ $=$ $\langle u^4, 2\cos(u^2), 2\sin(u^2)\rangle$, $0 \le u \le 2\pi$.

Comments: The right-hand curve is a re-parameterization of the left-hand curve ($u = \sqrt{t}$ or $t = u^2$). The right-hand parameterization traces out the curve from the left in $0 \le u \le \sqrt{2\pi}$. Since this right-hand curve goes farther, it has greater arc length.

(c) The arc length of the curve parameterized by $\mathbf{f}(t) = \langle 1+3t^2, 2-t^2, 5+2t^2\rangle$, $0 \le t \le 1$.

$<$

The arc length of the curve parameterized by $\mathbf{g}(u) = \langle\frac{1}{2}u^2, u, \frac{2\sqrt{2}}{3}u^{3/2}\rangle$, $0 \le u \le 2$.

Comments: The left-hand curve is disguised line segment from $(1, 2, 5)$ to $(4, 1, 7)$, which therefore has length $\sqrt{14}$. We can compute the arc length of the right-hand curve in the usual way: $\mathbf{g}'(u) = \langle u, 1, \sqrt{2u}\rangle$, so $|\mathbf{g}'(u)| = \sqrt{u^2 + 2u + 1} = u + 1$, so the right-hand arc length is

$$\int_0^2 |\mathbf{g}'(u)| \; du = \int_0^2 (u+1) \; du = 4.$$

Thus $\sqrt{14} < 4$.

(d) The arc length of the curve parameterized by $\mathbf{f}(t) = \langle\sin t, \cos t, t\rangle$, $1 \le t \le 5$.

$<$

The arc length of the curve parameterized by $\mathbf{g}(u) = \langle u\sin u, u\cos u, u\rangle$, $1 \le u \le 5$.

Solution: Both of these are helices from the $z = 1$ plane to the $z = 5$ plane. But the helix on the right-hand side is always wider than the helix on the left, so the arc length on the right is greater.

12 Find the distance between the cylinder $x^2 + y^2 = 1$ and the line L given by the symmetric equations

$$\frac{x+2}{4} = \frac{y-1}{3} = \frac{z}{2}.$$

Solution: The trick here is that the distance between the cylinder and the line is one less than the distance between the center line of the cylinder and the line L. (One less because the cylinder has radius 1.)

The line L is $\mathbf{r}(t) = \langle-2, 1, 0\rangle + t\langle 4, 3, 2\rangle$. It and the z-axis are skew (non-parallel, non-intersecting) lines, so we proceed with the usual trick. We find a plane that contains the z-axis and is parallel to L – then the distance from L to the z-axis is the distance from any single point on L to the plane.

The plane has normal perpendicular to the direction $\mathbf{k}$ of the z-axis and also perpendicular to the direction $\langle 4, 3, 2\rangle$ of L. Thus

$$\mathbf{n} = \mathbf{k} \times \langle 4, 3, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \langle-3, 4, 0\rangle.$$

Thus the plane is $-3x + 4y = 0$ (using the origin as a point on the plane).

We find the distance between the point $P(-2, 1, 0)$ on the line L and the plane by choosing a point $Q(0, 0, 0)$ on the plane.
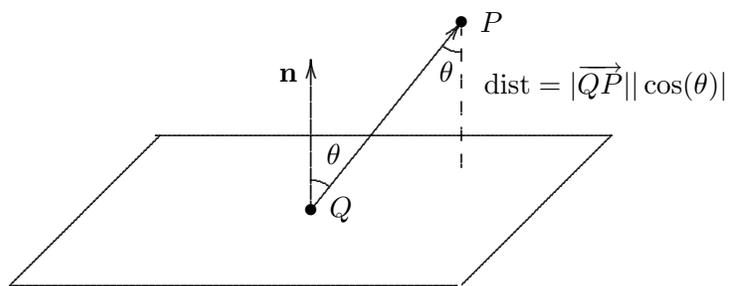
Hence $\overrightarrow{QP} = \langle -2, 1, 0 \rangle$. Using $\mathbf{n} = \langle -3, 4, 0 \rangle$, we get $\text{comp}_{\mathbf{n}} \overrightarrow{QP} = \frac{\mathbf{n} \cdot \overrightarrow{QP}}{|\mathbf{n}|} = \frac{10}{5} = 2$. Thus the distance between the line L and the plane – and hence between the two lines – is 2. Finally, the distance between the line L and the cylinder is $2 - 1 = 1$.