

PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

1 Which of the following statements are true?

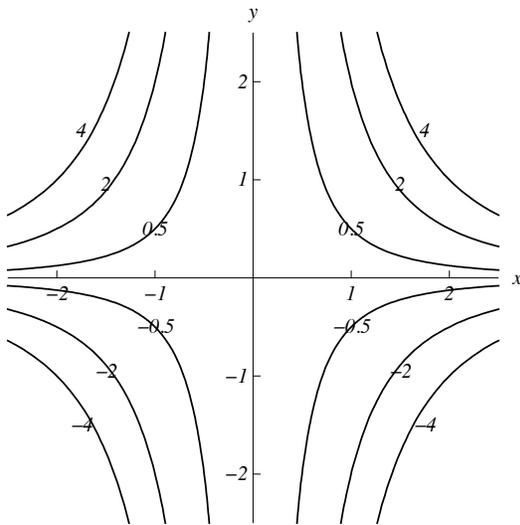
- (a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.
- (b) If $\mathbf{b} = \mathbf{c}$, then $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$.
- (c) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- (d) If $|\mathbf{a}| > |\mathbf{b}|$, then $\mathbf{a} \cdot \mathbf{c} > \mathbf{b} \cdot \mathbf{c}$.
- (e) If $|\mathbf{a}| = |\mathbf{b}|$, then either $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} = -\mathbf{b}$.
- (f) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.
- (g) $\mathbf{a} \times \mathbf{a} = |\mathbf{a}|^2$
- (h) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

Solution: (b) and (c) are true.

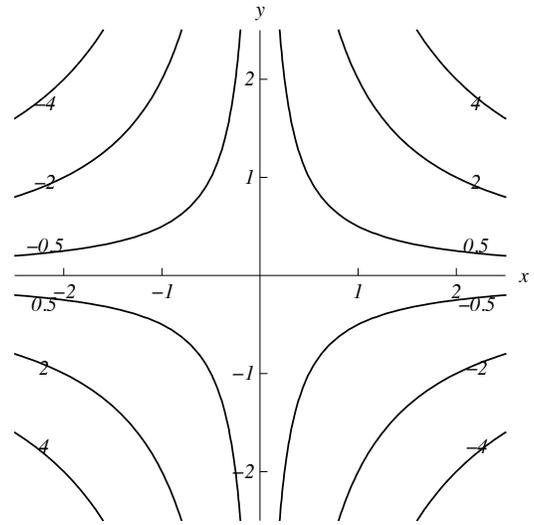
2 Match each contour plot with one of the following functions.

- | | |
|---------------------------|-------------------------|
| (a) $f(x, y) = x^2 - y^2$ | (e) $f(x, y) = x - y^2$ |
| (b) $f(x, y) = xy^2$ | (f) $f(x, y) = xy$ |
| (c) $f(x, y) = y^2 - x^2$ | (g) $f(x, y) = x^2y$ |
| (d) $f(x, y) = x^2y^2$ | (h) $f(x, y) = y - x^2$ |

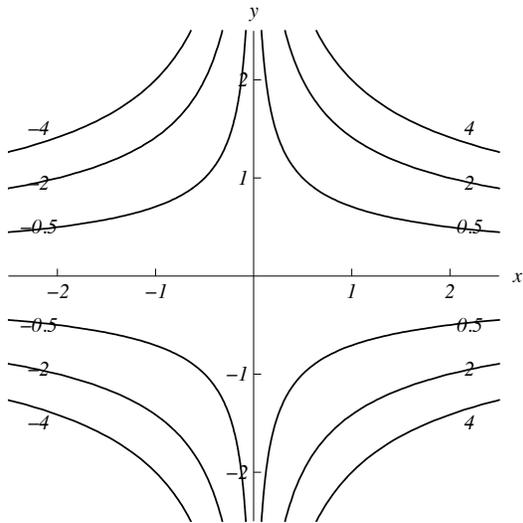
Solution:



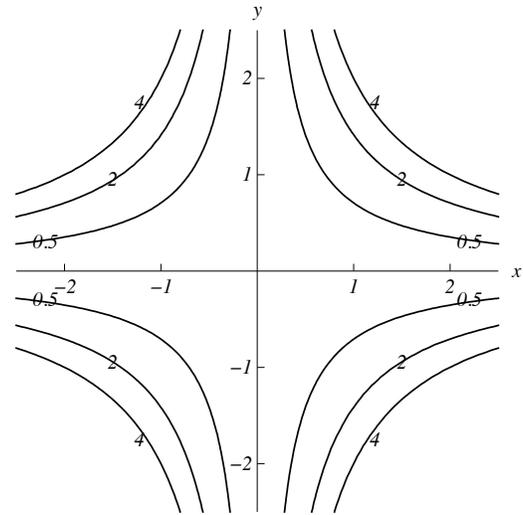
(i) is (g) $f(x, y) = x^2y$



(ii) is (f) $f(x, y) = xy$



(iii) is (b) $f(x, y) = xy^2$



(iv) is (d) $f(x, y) = x^2y^2$

3 Match each of the following equations with the appropriate graph below.

(a) $x^2 + 4y^2 + 9z^2 = 1$

(b) $x^2 - y^2 + z^2 = 1$

(c) $y = 2x^2 + z^2$

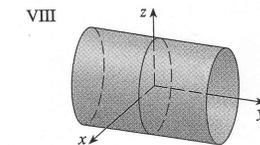
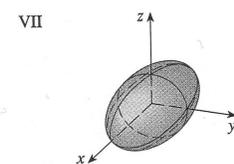
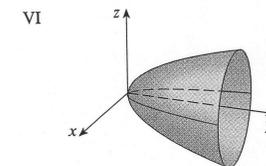
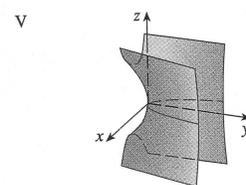
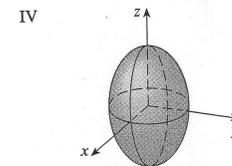
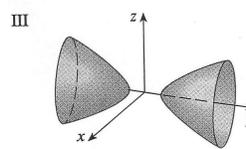
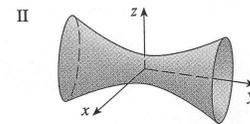
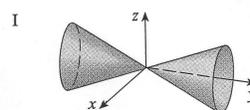
(d) $x^2 + 2z^2 = 1$

(e) $9x^2 + 4y^2 + z^2 = 1$

(f) $-x^2 + y^2 - z^2 = 1$

(g) $y^2 = x^2 + 2z^2$

(h) $y = x^2 - z^2$



Solution:

- | | | | |
|---------|---------|--------|----------|
| (a) VII | (b) II | (c) VI | (d) VIII |
| (e) IV | (f) III | (g) I | (h) V |

4 The volume of the parallelepiped determined by the vectors $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ is...

- (a) 54 (b) 36 (c) 108
 (d) 144 (e) None of the above.

Solution: The answer is (c). The volume is $|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}| = 108$.

5 Match each equation below with the corresponding equation in polar, spherical, or cylindrical coordinates.

- | | |
|---------------------------------|-------------------------------------|
| (a) $x^2 + y^2 + z^2 = x$ | (i) $r^2(1 - \sin 2\theta) = z^2$ |
| (b) $(x^2 + y^2)^2 = x^2 - y^2$ | (ii) $r^2 = z - 3$ |
| (c) $x^2 + y^2 = z - 3$ | (iii) $r^2 = \cos 2\theta$ |
| (d) $x^2 + y^2 - z^2 = 1$ | (iv) $\rho = \cos \theta \sin \phi$ |
| (e) $x^2 - 2xy + y^2 = z^2$ | (v) $\rho^2 \cos 2\phi = -1$ |

Solution:

- (a) (iv) (b) (iii) (c) (ii) (d) (v) (e) (i)

6 The function

$$f(x, y) = \frac{3x - 4y}{x^2 + y^2}.$$

satisfies which of the following partial differential equations.

- (a) $f_{xx} = f_{yy}$ (b) $f_x = f_{yy}$
 (c) $f_{xx} + f_{yy} = 0$ (d) $f_y = f_{xx}$
 (e) None of the above.

Solution: The correct answer is (c). Here are the first derivatives:

$$f_x = \frac{3(x^2 + y^2) - 2x(3x - 4y)}{(x^2 + y^2)^2} = \frac{-3x^2 + 8xy + 3y^2}{(x^2 + y^2)^2}$$

and

$$f_y = \frac{-4(x^2 + y^2) - 2y(3x - 4y)}{(x^2 + y^2)^2} = \frac{-4x^2 - 6xy + 4y^2}{(x^2 + y^2)^2}.$$

The second derivatives take slightly more work, but here are the final answers:

$$f_{xx} = \frac{6x^3 - 24x^2y - 18xy^2 + 8y^3}{(x^2 + y^2)^3}$$

$$f_{xy} = \frac{8x^3 + 18x^2y - 24xy^2 - 6y^3}{(x^2 + y^2)^3}$$

f_{yx} = same as f_{xy} by Clairaut's Theorem

$$f_{yy} = \frac{-6x^3 + 24x^2y + 18xy^2 - 8y^3}{(x^2 + y^2)^3}$$

7 Assume that \mathbf{u} and \mathbf{v} are *unit* vectors. For each of the following assumptions, determine whether the two vectors must be (i) perpendicular, (ii) pointing in the same direction, (iii) pointing in opposite directions, or (iv) parallel but one cannot tell from the given information whether they point in the same or opposite directions:

(a) $\mathbf{u} \cdot \mathbf{v} = 1$

Solution: \mathbf{u} and \mathbf{v} point in the same direction (ii).

(b) $\mathbf{u} \cdot \mathbf{v} = 0$

Solution: \mathbf{u} and \mathbf{v} are perpendicular (i).

(c) $\mathbf{u} \cdot \mathbf{v} = -1$

Solution: \mathbf{u} and \mathbf{v} point in opposite directions (iii).

(d) $|\mathbf{u} \cdot \mathbf{v}| = 1$

Solution: \mathbf{u} and \mathbf{v} are parallel but one cannot tell from the given information whether they point in the same or opposite directions (iv).

(e) $|\mathbf{u} \cdot \mathbf{v}| = 0$

Solution: \mathbf{u} and \mathbf{v} are perpendicular (i).

(f) $|\mathbf{u} \times \mathbf{v}| = 1$

Solution: \mathbf{u} and \mathbf{v} are perpendicular (i).

(g) $|\mathbf{u} \times \mathbf{v}| = 0$

Solution: \mathbf{u} and \mathbf{v} are parallel but one cannot tell from the given information whether they point in the same or opposite directions (iv).

PART II: Free response questions. You should attempt all parts of each problem. Show your work!

8 Let

$$A = (1, 0, 0) \quad B = (1, 1, 2) \quad C = (1, 2, 1) \quad D = (2, 2, 3).$$

Find

(a) The distance from D to A .

Solution: The distance from D to A is

$$\sqrt{(1-2)^2 + (0-2)^2 + (0-3)^2} = \sqrt{14}.$$

(b) The distance from D to the line containing A and B .

Solution: Let $\mathbf{a} = \langle 0, 1, 2 \rangle$ be the vector from A to B and $\mathbf{b} = \langle 1, 2, 3 \rangle$ be the vector from A to D . Then

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \left\langle 0, \frac{8}{5}, \frac{16}{5} \right\rangle.$$

The distance from D to the line containing A and B is

$$|\mathbf{b} - \text{proj}_{\mathbf{a}} \mathbf{b}| = \frac{\sqrt{30}}{5}.$$

Another approach: using the same terminology as above, the distance from D to the line is $|\mathbf{b}| \sin(\theta)$, where θ is the angle between \mathbf{a} and \mathbf{b} . Since $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta)$, we can use the cross product to compute the distance. Notice that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \langle -1, 2, -1 \rangle,$$

so the distance is

$$|\mathbf{b}| \sin(\theta) = \frac{|\mathbf{a}| |\mathbf{b}| \sin(\theta)}{|\mathbf{a}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle -1, 2, -1 \rangle|}{|\langle 0, 1, 2 \rangle|} = \frac{\sqrt{6}}{\sqrt{5}} = \frac{\sqrt{30}}{5},$$

which is the same as before.

- (c) An equation of the plane containing A , B , and C .

Solution: Let $\mathbf{a} = \langle 0, 1, 2 \rangle$ be the vector from A to B and $\mathbf{c} = \langle 0, 2, 1 \rangle$ be the vector from A to C . Then a vector perpendicular to the plane is $\mathbf{a} \times \mathbf{c} = \langle -3, 0, 0 \rangle$. Thus, the equation of the plane is

$$\langle -3, 0, 0 \rangle \cdot \langle x - 1, y, z \rangle = 0 \quad \text{or} \quad x = 1.$$

- 9 An astronaut is flying in a spacecraft along the path described by

$$\mathbf{r}(t) = \langle t^2 - t, 2 + t, -\frac{3}{t} \rangle,$$

where t is given in hours.

- (a) What is the velocity of the spacecraft when it reaches the point $(6, 5, -1)$?

Solution: The spacecraft reaches the point $(6, 5, -1)$, when $t = 3$. Since $\mathbf{r}'(t) = \langle 2t - 1, 1, \frac{3}{t^2} \rangle$, the velocity at $t = 3$ is $\mathbf{r}'(3) = \langle 5, 1, \frac{1}{3} \rangle$.

- (b) What is the speed of the spacecraft when it reaches the point $(6, 5, -1)$?

Solution: The speed at $t = 3$ is $|\mathbf{r}'(3)| = \frac{\sqrt{235}}{3}$.

- (c) What is the acceleration of the spacecraft when it reaches the point $(6, 5, -1)$?

Solution: Since $\mathbf{r}''(t) = \langle 2, 0, -\frac{6}{t^3} \rangle$, the acceleration at $t = 3$ is $\mathbf{r}''(3) = \langle 2, 0, -\frac{2}{9} \rangle$.

- (d) If the engines of the spacecraft are shut off when it reaches the point $(6, 5, -1)$, where will the spacecraft be 2 hours later?

Solution: The spacecraft will continue in a straight line $(6, 5, -1)$ in the direction of the velocity vector $\mathbf{r}'(3) = \langle 5, 1, \frac{1}{3} \rangle$. Its position at time t is given by

$$\mathbf{p}(t) = \langle 6, 5, -1 \rangle + t \langle 5, 1, \frac{1}{3} \rangle$$

and $\mathbf{p}(2) = \langle 16, 7, -\frac{1}{3} \rangle$.

- 10 Consider the parameterized curve $\mathbf{r}(t) = \langle 12t, 5 \cos t, 3 - 5 \sin t \rangle$. Find the arc length from $t = 0$ to $t = 2$.

Solution: First, $\mathbf{r}'(t) = \langle 12, -5 \sin t, -5 \cos t \rangle$. Thus,

$$\begin{aligned} \int_0^2 |\mathbf{r}'(t)| dt &= \int_0^2 \sqrt{144 + 25 \sin^2 t + 25 \cos^2 t} dt \\ &= \int_0^2 13 dt = 26. \end{aligned}$$

- 11 Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = (2t + 1)\mathbf{i} + \cos t\mathbf{j} - e^t\mathbf{k}$ and $\mathbf{r}(0) = -\mathbf{i} + \pi\mathbf{j} + 3\mathbf{k}$.

Solution:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t) dt \\ &= \int (2t + 1)\mathbf{i} + \cos t\mathbf{j} - e^t\mathbf{k} dt \\ &= (t^2 + t)\mathbf{i} + \sin t\mathbf{j} - e^t\mathbf{k} + \mathbf{C}\end{aligned}$$

Since $\mathbf{r}(0) = -\mathbf{k} + \mathbf{C} = -\mathbf{i} + \pi\mathbf{j} + 3\mathbf{k}$, we have $\mathbf{r}(t) = (t^2 + t - 1)\mathbf{i} + \sin t + \pi\mathbf{j} + (4 - e^t)\mathbf{k}$.

- 12 (a) Find a parameterization of the line of intersection of the planes $3x - 2y + z = 7$ and $x + 2y + 3z = -3$.

Solution: Any vector in the direction of the line is perpendicular to both normals. One such vector is

$$\langle 3, -2, 1 \rangle \times \langle 1, 2, 3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle -8, -8, 8 \rangle.$$

Another one is $\langle 1, 1, -1 \rangle$. (Why?) A point on both planes (and thus the line) is $(1, -2, 0)$. (To find this, I added the equations to see that any point on the line satisfies $4x + 4y = 4$. Then I picked a nice z – namely $z = 0$ – and found $x = 1$ and also $y = -2$.) Thus the equation of the line is

$$\mathbf{r}(t) = \langle 1, -2, 0 \rangle + t\langle 1, 1, -1 \rangle.$$

- (b) Find the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

representing this line.

Solution: This is

$$\frac{x - 1}{1} = \frac{y + 2}{1} = \frac{z}{-1} \quad \text{or simply} \quad x - 1 = y + 2 = -z.$$

- 13 The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

- (a) Parameterize each curve in the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Solution: Both these surfaces are elliptical cylinders. The intersection curves, when translated vertically into the xy -plane, each project onto the ellipse $x^2 + \frac{y^2}{2} = 1$. This we can view as $(x)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$ and take the parameterization $x = \cos t$ and $\frac{y}{\sqrt{2}} = \sin t$. What about z ? Well, from the equations it's clear that $z^2 = x^2$, so $z = x$ or $z = -x$ (hence two curves). So our parameterizations are

$$\mathbf{r}(t) = \langle \cos t, \sqrt{2} \sin t, \cos t \rangle \quad \text{and} \quad \mathbf{r}(t) = \langle \cos t, \sqrt{2} \sin t, -\cos t \rangle.$$

- (b) Set up the integral for the arc length of one of the curves.

Solution: We'll work with the first one. Notice that $\mathbf{r}'(t) = \langle -\sin t, \sqrt{2}\cos t, -\sin t \rangle$, so $|\mathbf{r}'(t)| = \sqrt{2}$. Thus the arc length is given by the integral

$$\int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{2} dt.$$

(c) What is the arc length of this curve?

Solution: This integral is $2\pi\sqrt{2}$.

14 Imagine the planet Earth as the unit sphere centered at the origin in three-dimensional space. An asteroid is approaching from the point $P(0, 4, 3)$ along the path

$$\mathbf{r}(t) = \langle (4-t)\sin(t), (4-t)\cos(t), 3-t \rangle.$$

(a) When and where will it first hit the earth?

Solution: The asteroid will hit the earth when $x^2 + y^2 + z^2 = 1$. In terms of t , this is when

$$(4-t)^2 \sin^2 t + (4-t)^2 \cos^2 t + (3-t)^2 = 1 \quad \text{or} \quad (4-t)^2 + (3-t)^2 = 1 \quad \text{or} \quad 2t^2 - 14t + 24 = 0.$$

This factors to $2(t-3)(t-4) = 0$, so $t = 3$ and $t = 4$. Thus the asteroid will *first* hit the earth when $t = 3$ at $\mathbf{r}(3) = \langle \sin(3), \cos(3), 0 \rangle$.

(b) What velocity will it have at impact?

Solution: The velocity will be $\mathbf{r}'(3)$. We compute the derivative to be

$$\mathbf{r}'(t) = \langle (4-t)\cos t - \sin t, (t-4)\sin t - \cos t, -1 \rangle,$$

so $\mathbf{r}'(3) = \langle \cos(3) - \sin(3), -\sin(3) - \cos(3), -1 \rangle$ is the velocity. (The speed, by the way, will be a nice simple $|\mathbf{r}'(3)| = \sqrt{3}$.)