PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

1 Fill in the boxes. No additional explanations required.

| | | Answer |
|---|---|---|
| Chain rule | $\frac{d}{dt}f(\mathbf{r}(t)) = \square \cdot \mathbf{r}'(t)$ | $\nabla f$ |
| Directional derivative | $D_{\langle 1,1\rangle/\sqrt{2}}f(1,1) = \nabla f(1,1) \cdot \square$ | $\langle 1,1\rangle/\sqrt{2}$ |
| Linearization of $f(x,y)$ at $(1,1)$ | $L(x,y) = \square + \nabla f(1,1) \cdot \langle x-1, y-1\rangle$ | $f(1,1)$ |
| Equation of tangent line at $(1,1)$ | $\nabla f(1,1) \cdot \langle x-1, y-1\rangle = \square$ | 0 |
| Critical point $(1,1)$ of $f$ | $\nabla f(1,1) = \square$ | $\langle 0,0\rangle$ or $\mathbf{0}$ |
| Lagrange equations | $\nabla f(x,y) = \square \nabla g(x,y),\ g(x,y) = c$ | $\lambda$ |
| Type I Integral | $\int_a^b \int_{c(x)}^{d(x)} f(x,y) \square$ | $dy\ dx$ |
| Type II Integral | $\int_c^d \int_{a(y)}^{b(y)} f(x,y) \square$ | $dx\ dy$ |
| Integral in polar coordinates | $\int_a^b \int_{f(\theta)}^{g(\theta)} \square f(r\cos(\theta), r\sin(\theta))\ dr\ d\theta$ | $r$ |
| Area | $\iint_R \square\ dx\ dy$ | 1 |

2 Each of the following functions has a critical point at the origin $(0,0)$. For each function, what can you conclude about the nature of this critical point by applying the Second Derivative Test? Your answer should be one of *local maximum*, *local minimum*, *saddle*, or *inconclusive*.

(a) $f(x,y) = x^2 - 4xy + 2y^2$

(b) $f(x,y) = x^4 + 2x^2y^2 + x^3$

(c) $f(x,y) = x^2 + 2y^2$

(d) $f(x,y) = -x^2 + xy - y^2$

**Solution:**

(a) Since $f_{xx}f_{yy} - f_{xy}^2 = -8$, the origin is a saddle.

(b) Since $f_{xx}f_{yy} - f_{xy}^2 = 0$, the Second Derivative Test is inconclusive.

(c) Since $f_{xx}f_{yy} - f_{xy}^2 = 8$ and $f_{xx} = 2 > 0$, the origin is a minimum.

(d) Since $f_{xx}f_{yy} - f_{xy}^2 = 3$ and $f_{xx} = -2 < 0$, the origin is a maximum.

3 If we change the order of integration of the integral

$$\int_1^4 \int_0^{\ln y} f(x,y)\ dx\ dy,$$

which of the following integrals do we obtain?

(a) $\int_0^{\ln y} \int_1^4 f(x,y)\ dy\ dx$

(b) $\int_0^{\ln 4} \int_4^{e^x} f(x,y)\ dy\ dx$

(c) $\int_0^4 \int_{e^x}^4 f(x,y)\ dy\ dx$

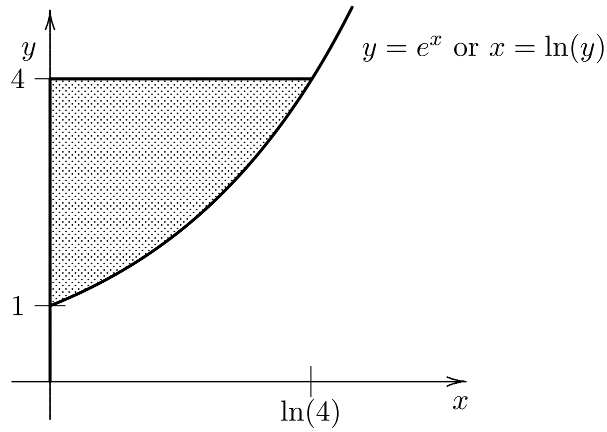
(d) $\int_0^4 \int_{e^x}^{\ln 4} f(x,y)\ dy\ dx$

(e) None of the above.

**Solution:** Since

$$\int_1^4 \int_0^{\ln y} f(x,y)\,dx\,dy = \int_0^{\ln 4} \int_{e^x}^4 f(x,y)\,dy\,dx,$$

the correct solution is none of the above (e). Here is a small sketch with the region of integration shaded:




4 If

$$x^2 + y^2 z + xz^4 + xyz^7 = 0,$$

then which of the following is $\partial z/\partial x$ at the point $(x, y, z) = (1, 0, -1)$.

(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) $-\frac{4}{3}$ (e) None of the above.

**Solution:** Let

$$F(x,y,z) = x^2 + y^2 z + xz^4 + xyz^7.$$

Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{2x + z^4 + yz^7}{y^2 + 4xz^3 + 7xyz^6}.$$

Therefore,

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(1,0,-1)} = \frac{3}{4},$$

and the correct answer is (a).

5 Let $z = f(x,y)$ have the contour plot shown in Figure 1 below, where the horizontal axis is the $x$-axis and the vertical axis is the $y$-axis. At which of the labelled points below is the directional derivative in the direction $\mathbf{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ the greatest?

(a) $A$ (b) $B$ (c) $C$ (d) $D$

**Solution:** At the point $B$, moving in the direction $\mathbf{u}$ (which I'll call "northwest") decreases the function $f$. The same thing is happening at point $D$ (moving from the level curve $f = 4$ toward the level curve $f = 2$). At $A$ we're moving from $f = 4$ almost immediately back to $f = 4$. But at $C$, we're increasing quickly to $f = 6$ and $f = 8$. Thus the correct answer is (c).

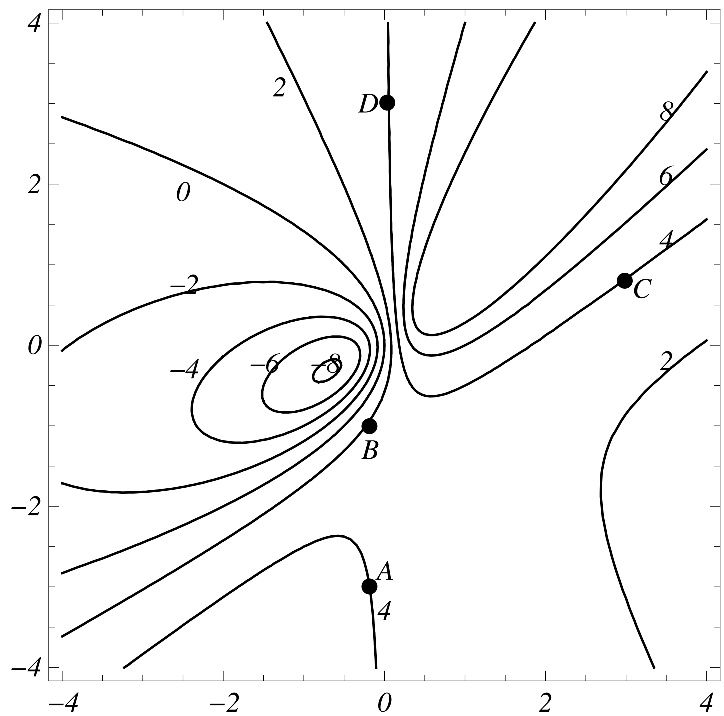
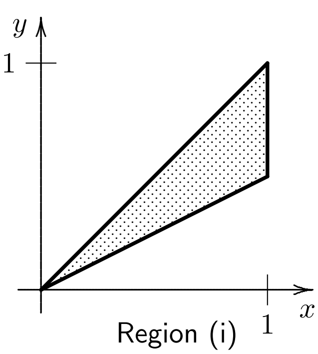
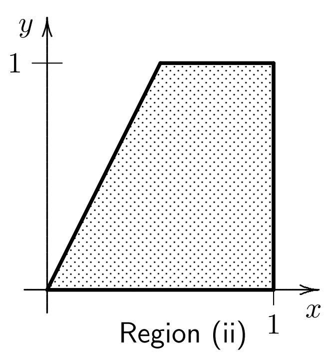
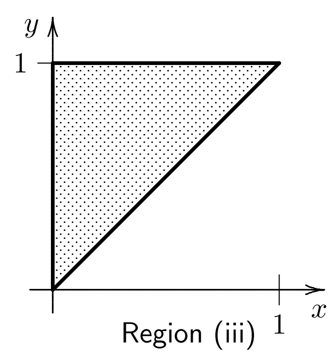


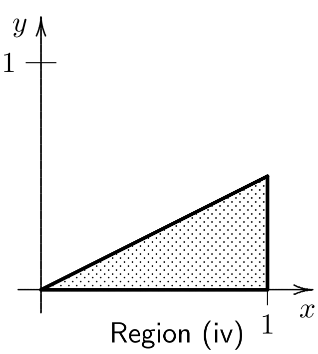

Figure 1: Figure For Problem 5.

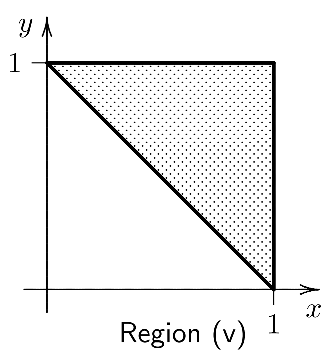
6 Match the regions with the corresponding double integrals:

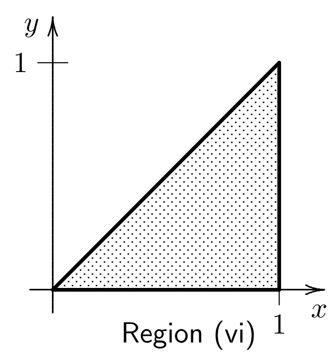



(a) $\int_0^1 \int_{x/2}^{x} f(x,y)\, dy\, dx$ (b) $\int_0^1 \int_0^{y} f(x,y)\, dx\, dy$ (c) $\int_0^1 \int_0^{x/2} f(x,y)\, dy\, dx$

(d) $\int_0^1 \int_{y/2}^{1} f(x,y)\, dx\, dy$ (e) $\int_0^1 \int_0^{x} f(x,y)\, dy\, dx$ (f) $\int_0^1 \int_{1-x}^{1} f(x,y)\, dy\, dx$

**Solution:**

(a) This region is

$$D = \left\{(x, y) \mid \tfrac{x}{2} \leq y \leq x,\ 0 \leq y \leq 1\right\}.$$

This corresponds to Region (i).

(b) This region is $D = \{(x, y) \mid 0 \leq x \leq y,\ 0 \leq y \leq 1\}$, or Region (iii).

(c) This region is $D = \left\{(x, y) \mid 0 \leq y \leq \frac{x}{2},\ 0 \leq y \leq 1\right\}$, or Region (iv).

(d) This region is $D = \left\{(x, y) \mid \frac{y}{2} \leq x \leq 1,\ 0 \leq x \leq 1\right\}$, or Region (ii).

(e) This region is $D = \{(x, y) \mid 0 \leq y \leq x,\ 0 \leq x \leq 1\}$, or Region (vi).

(f) This region is $D = \{(x, y) \mid 1 - x \leq y \leq 1,\ 0 \leq x \leq 1\}$, or Region (v).

7 Answer the following questions True or False:

(a) **T F** The directional derivative $D_{\mathbf{v}} f$ is a vector perpendicular to $\mathbf{v}$.

**Solution:** This is **False**. The directional derivative $D_{\mathbf{v}} f$ is the scalar $\nabla f \cdot \mathbf{v}$.

(b) **T F** Given a curve $\mathbf{r}(t)$ on a surface $g(x, y, z) = 1$, then $\frac{d}{dt} g(\mathbf{r}(t)) = 0$.

**Solution:** This is **True**. This is simply the chain rule applied to the equation $g(\mathbf{r}(t)) = 1$.

(c) **T F** If $f(x, y)$ has a local maximum at $(0, 0)$, then it is possible that $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) < 0$.

**Solution:** This is **False**. This is saying that $f$ is concave up in the $x$ direction (in the $xz$-plane) and concave down in the $y$ direction (in the $yz$-plane). This cannot then be a local maximum.

(d) **T F** Fubini's theorem assures us that $\int_0^1 \int_0^x f(x, y)\, dy\, dx = \int_0^1 \int_0^y f(x, y)\, dx\, dy$.

This is **False**. The two regions described by the iterated integrals are not the same – one is Region (iii) of Problem 6 and the other is Region (vi).

(e) **T F** If $x + \sin(xy) = 1$, then $\frac{dy}{dx} = -\frac{1 + y\cos(xy)}{x\cos(xy)}$.

**Solution:** This is **True**.

(f) **T F** The directional derivative $D_{\mathbf{v}} f(1, 1)$ is zero if $\mathbf{v}$ is a unit vector tangent to the level curve of $f$ which passes through the point $(1, 1)$.

**Solution:** This is **True**. This is our most important fact about the gradient: it is perpendicular to the tangent line (or plane) of the level curve (or surface).

## Part II: Free response questions. You should attempt all parts of each problem. Show your work!

8 Let $f(x, y) = y^2 - x^2$.

(a) Calculate the gradient of $f$, $\nabla f$.

**Solution:** $\nabla f(x, y) = -2x\,\mathbf{i} + 2y\,\mathbf{j}$ or $\langle -2x, 2y\rangle$.

(b) What is the directional derivative of $f$ in the direction $\mathbf{v} = -3\,\mathbf{i} + 4\,\mathbf{j}$ at the point $(1, 1)$?

**Solution:** A unit vector in the direct of $\mathbf{v}$ is $\mathbf{u} = -(3/5)\,\mathbf{i} + (4/5)\,\mathbf{j}$. Thus,

$$D_{\mathbf{u}} f = \nabla f(1, 1) \cdot \mathbf{u} = \langle -2, 2\rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5}\right\rangle = \frac{14}{5}.$$

(c) Find the maximum and minimum values of $f(x, y) = y^2 - x^2$ subject to the constraint $g(x, y) = x^2 + 4y^2 = 4$.

**Solution:** The gradient of $g$ is $\nabla g = \langle 2x, 8y\rangle$. If $\nabla f = \lambda \nabla g$, then

$$\begin{aligned} -2x &= \lambda 2x \\ 2y &= \lambda 8y \\ x^2 + 4y^2 &= 4. \end{aligned}$$

By the last equation, $x$ and $y$ cannot both be zero. If $x \neq 0$, then $\lambda = -1$ and $y = 0$. Thus, we have critical points at $(\pm 2, 0)$. If $y \neq 0$, then $\lambda = 1/4$ and $x = 0$. In this case we have critical points at $(0, \pm 1)$. Therefore, $f(2,0) = f(-2,0) = -4$ is the minimum value and $f(1,0) = f(-1,0) = 1$ is the maximum value.

9 (a) Locate and classify all the critical points of

$$f(x,y) = 3y - y^3 - 3x^2y.$$

**Solution:** The critical points of $f(x,y)$ are those points where $\nabla f = \mathbf{0}$. Since $\nabla f = \langle -6xy, 3 - 3y^2 - 3x^2 \rangle$, the critical points are where $xy = 0$ and $x^2 + y^2 = 1$. This gives us four points: $(0, \pm 1)$ and $(\pm 1, 0)$.

We can compute $f_{xx} = -6y$, $f_{xy} = f_{yx} = -6x$, and $f_{yy} = -6y$. Thus $D = 36y^2 - 36x^2$, and we get a small chart:

| Crit. Pt. | $D$ | $f_{xx}$ | Classification | $f(x,y)$ |
|---|---|---|---|---|
| $(0,1)$ | 36 | $-6$ | Maximum | 2 |
| $(0,-1)$ | 36 | 6 | Minimum | $-2$ |
| $(1,0)$ | $-36$ | 0 | Saddle | 0 |
| $(-1,0)$ | $-36$ | 0 | Saddle | 0 |

(b) Where on the parameterized surface

$$\mathbf{r}(x,y) = \langle u, v, w \rangle = \left\langle xy^3, \frac{x^2}{2}, \frac{3y^2}{2} \right\rangle$$

is the function $g(u,v,w) = u - v - w$ extremal? To investigate this, find all the critical points of the function $f(x,y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum, or a saddle point and show how you know.

**Solution:** The critical points of $f(x,y)$ are again those points where $\nabla f = \mathbf{0}$. In this case $\nabla f = \langle y^3 - x, 3xy^2 - 3y \rangle$, so the critical points are where $x = y^3$ and $3y(xy - 1) = 0$. Thus either $y = 0$ (so $x = 0$ as well) or $xy = 1$, in which case $x^4 = 1$. This gives us three points: $(0,0)$, $(1,1)$, and $(-1,-1)$.

We compute $f_{xx} = -1$, $f_{xy} = f_{yx} = 3y^2$, and $f_{yy} = 6xy - 3$. Thus $D = 3 - 6xy - 9y^4$, and we again get a small chart:

| Crit. Pt. | $D$ | $f_{xx}$ | Classification | $f(x,y)$ |
|---|---|---|---|---|
| $(0,0)$ | 3 | $-1$ | Maximum | 0 |
| $(1,1)$ | $-12$ | $-1$ | Saddle | $-1$ |
| $(-1,-1)$ | $-12$ | $-1$ | Saddle | $-1$ |

10 Find an equation of the tangent plane to the surface $x^2y + e^{xz} + yz = 3$ at the point $(0,1,2)$.

**Solution:** Let $F(x,y,z) = x^2y + e^{xz} + yz$. Then $F(x,y,z) = 3$ is a level surface for $F$. Since

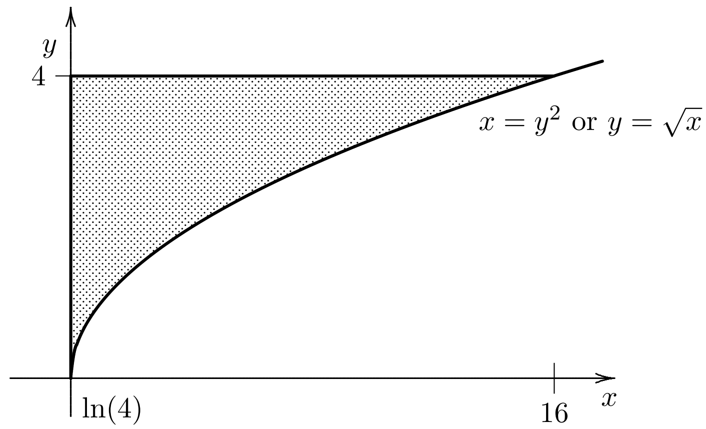
$$\nabla F(x,y,z) = \langle 2xy + ze^{xz}, x^2 + z, xe^{xz} + y \rangle,$$

$\nabla F(0,1,2) = \langle 2,2,1 \rangle$ is a normal vector to the surface at $(0,1,2)$. Therefore, the tangent plane is given by

$$\langle 2,2,1 \rangle \cdot \langle x - 0, y - 1, z - 2 \rangle = 0 \qquad \text{or} \qquad 2x + 2(y-1) + (z-2) = 0.$$

11 Evaluate the double integral $\int_0^4 \int_0^{y^2} \frac{x^4}{4 - \sqrt{x}}\, dx\, dy$.

**Solution:** This is too difficult to integrate directly, so we switch the order of integration. This is easier if we have a small sketch of the region of integration (shaded):




We can then reverse the order of integration and integrate:

$$\int_0^4 \int_0^{y^2} \frac{x^4}{4-\sqrt{x}}\,dx\,dy = \int_0^{16}\int_{\sqrt{x}}^4 \frac{x^4}{4-\sqrt{x}}\,dy\,dx$$
$$= \int_0^{16} \left( \frac{x^4}{4-\sqrt{x}} y\Big|_{\sqrt{x}}^4 \right) dx = \int_0^{16} \left( \frac{x^4}{4-\sqrt{x}}(4-\sqrt{x}) \right) dx$$
$$= \int_0^{16} x^4\,dx$$
$$= \frac{1}{5}x^5\Big|_0^{16} = \frac{16^5}{5}.$$

12 Let $g(x, y, z) = x^2 + 2y^2 - z - 3$.

(a) Find the equation of the tangent plane to the level surface $g(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (2, 0, 1)$.

**Solution:** The tangent plane to this level surface has normal $\nabla g(2, 0, 1)$. Since $\nabla g = \langle 2x, 4y, -1\rangle$, this normal is $\langle 4, 0, -1\rangle$. Thus the tangent plane has equation

$$\langle 4, 0, -1\rangle \cdot x - 2, y - 0, z - 1\rangle = 0 \qquad \text{or} \qquad 4x - z = 7.$$

(b) The surface in part (a) is the graph $z = f(x, y)$ of a function of two variables. Find the tangent line to the level curve $f(x, y) = 1$ at the point $(x_0, y_0) = (2, 0)$.

**Solution:** We can solve for $z$ and find $z = f(x, y) = x^2 + 2y^2 - 3$. Thus $\nabla f = \langle 2x, 4y\rangle$, which at the point $(2, 0)$ is $\langle 4, 0\rangle$. Finding the tangent line is then very similar to finding the tangent plane in part (a):

$$\langle 4, 0\rangle \cdot x - 2, y - 0\rangle = 0 \qquad \text{or} \qquad 4x = 8 \qquad \text{or} \qquad x = 2.$$

13 (a) Use the technique of linear approximation to estimate $f(\frac{\pi}{2} + 0.1, 2.9)$ for

$$f(x, y) = \left(10\sin(x) - 5y^2 + 8\right)^{1/3}.$$

**Solution:** Near the point $(\frac{\pi}{2}, 3)$, we have the linear approximation

$$f(x, y) \approx L(x, y) = f(\tfrac{\pi}{2}, 3) + f_x(\tfrac{\pi}{2}, 3)(x - \tfrac{\pi}{2}) + f_y(\tfrac{\pi}{2}, 3)(y - 3).$$

Since

$$f_x = \tfrac{1}{3}\left(10\sin(x) - 5y^2 + 8\right)^{-2/3} 10\cos(x) = \frac{10\cos(x)}{3(10\sin(x) - 5y^2 + 8)^{2/3}}$$

and

$$f_y = \tfrac{1}{3}\left(10\sin(x) - 5y^2 + 8\right)^{-2/3} (-10y) = -\frac{10y}{3(10\sin(x) - 5y^2 + 8)^{2/3}},$$

we get $f(\frac{\pi}{2}, 3) = -3$, $f_x(\frac{\pi}{2}, 3) = 0$, and $f_y(\frac{\pi}{2}, 3) = -\frac{10}{9}$. Thus

$$f(x, y) \approx -3 + 0(x - \tfrac{\pi}{2}) - \frac{10}{9}(y - 3)$$

and so

$$f(\tfrac{\pi}{2} + 0.1, 2.9) \approx -3 + 0(0.1) - \frac{10}{9}(-0.1) = \frac{26}{9}.$$

(b) Find the unit vector at $(\frac{\pi}{2}, 3)$ in the direction where this function increases fastest.

**Solution:** This function increases fastest in the $\nabla f$ direction. Since $\nabla f(\frac{\pi}{2}, 3) = \langle 0, -\frac{10}{9}\rangle$, the unit vector in this direction is $\langle 0, -1\rangle$.

14 A solid cone of height $h$ and with base radius $r$ has volume $f(h, r) = \frac{1}{3}\pi h r^2$ and surface area $g(h, r) = \pi r\sqrt{r^2 + h^2} + \pi r^2$. Among all cones with fixed surface area $g(h, r) = \pi$, find the cone with maximal volume using the method of Lagrange multipliers.

**Solution:** We're trying to maximize the function $f(h, r) = \frac{1}{3}\pi h r^2$ subject to the constraint $\pi r\sqrt{r^2 + h^2} + \pi r^2 = \pi$. We'll simplify this constraint to $g(r, h) = r\sqrt{r^2 + h^2} + r^2 = 1$. The equations $\nabla f = \lambda \nabla g$ gives

$$\tfrac{2}{3}\pi h r = \lambda\left(\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r\right)$$
$$\tfrac{1}{3}\pi r^2 = \lambda\left(\frac{rh}{\sqrt{r^2 + h^2}}\right).$$

If we assume that neither $h$ nor $r$ is zero (reasonable, since these both give cones with zero volume), then we can solve for $\lambda$ in the second equation: $\lambda = \frac{\pi r\sqrt{r^2 + h^2}}{3h}$. Plugging this into the first equation yields

$$\tfrac{2}{3}\pi h r = \frac{\pi r\sqrt{r^2 + h^2}}{3h}\left(\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r\right)$$

or (after multiplying through by $\frac{3h}{\pi r}$)

$$2h^2 = r^2 + h^2 + r^2 + 2r\sqrt{r^2 + h^2} \qquad \text{or} \qquad \sqrt{r^2 + h^2} = \frac{h^2 - 2r^2}{2r}.$$

If we plug this expression into our constraint equation, we get $h^2 = 2$ or $h = \sqrt{2}$ (we can ignore the negative square root as $h$ is a height). From the constraint we now see that $r = \frac{1}{2}$, so the maximum volume is $f(\sqrt{2}, \frac{1}{2}) = \frac{\pi\sqrt{2}}{12}$.