PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

1 Fill in the boxes. No additional explanations required.

Chain rule	$\frac{d}{dt}f(\mathbf{r}(t)) = \boxed{} \cdot \mathbf{r}'(t)$
Directional derivative	$D_{\langle 1,1\rangle/\sqrt{2}}f(1,1) = \nabla f(1,1) \cdot \boxed{}$
Linearization of $f(x,y)$ at $(1,1)$	$L(x,y) = \boxed{} + \nabla f(1,1) \cdot \langle x-1, y-1\rangle$
Equation of tangent line at $(1,1)$	$\nabla f(1,1) \cdot \langle x-1, y-1\rangle = \boxed{}$
Critical point $(1,1)$ of f	$\nabla f(1,1) = \boxed{}$
Lagrange equations	$\nabla f(x,y) = \boxed{} \nabla g(x,y),\ g(x,y) = c$
Type I Integral	$\int_a^b \int_{c(x)}^{d(x)} f(x,y) \boxed{}$
Type II Integral	$\int_c^d \int_{a(y)}^{b(y)} f(x,y) \boxed{}$
Integral in polar coordinates	$\int_a^b \int_{f(\theta)}^{g(\theta)} \boxed{} f(r\cos(\theta), r\sin(\theta))\ dr\ d\theta$
Area	$\iint_R \boxed{}\ dx\ dy$

2 Each of the following functions has a critical point at the origin $(0,0)$. For each function, what can you conclude about the nature of this critical point by applying the Second Derivative Test? Your answer should be one of *local maximum*, *local minimum*, *saddle*, or *inconclusive*.

(a) $f(x,y) = x^2 - 4xy + 2y^2$

(b) $f(x,y) = x^4 + 2x^2y^2 + x^3$

(c) $f(x,y) = x^2 + 2y^2$

(d) $f(x,y) = -x^2 + xy - y^2$

3 If we change the order of integration of the integral

$$\int_1^4 \int_0^{\ln y} f(x,y)\ dx\ dy,$$

which of the following integrals do we obtain?

(a) $\displaystyle\int_0^{\ln y} \int_1^4 f(x,y)\ dy\ dx$

(b) $\displaystyle\int_0^{\ln 4} \int_4^{e^x} f(x,y)\ dy\ dx$

(c) $\displaystyle\int_0^4 \int_{e^x}^4 f(x,y)\ dy\ dx$

(d) $\displaystyle\int_0^4 \int_{e^x}^{\ln 4} f(x,y)\ dy\ dx$

(e) None of the above.

4 If

$$x^2 + y^2z + xz^4 + xyz^7 = 0,$$

then which of the following is $\partial z/\partial x$ at the point $(x,y,z) = (1,0,-1)$.

(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) $-\frac{4}{3}$ (e) None of the above.

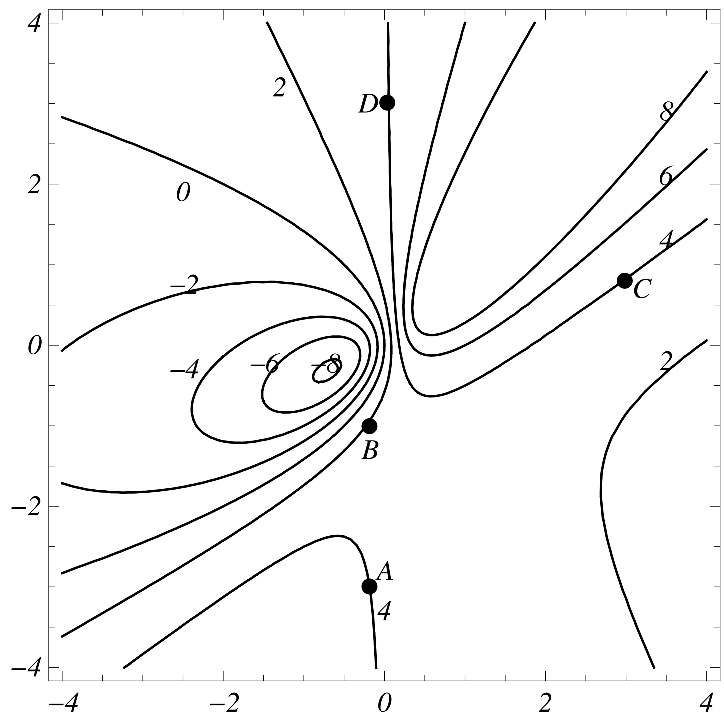
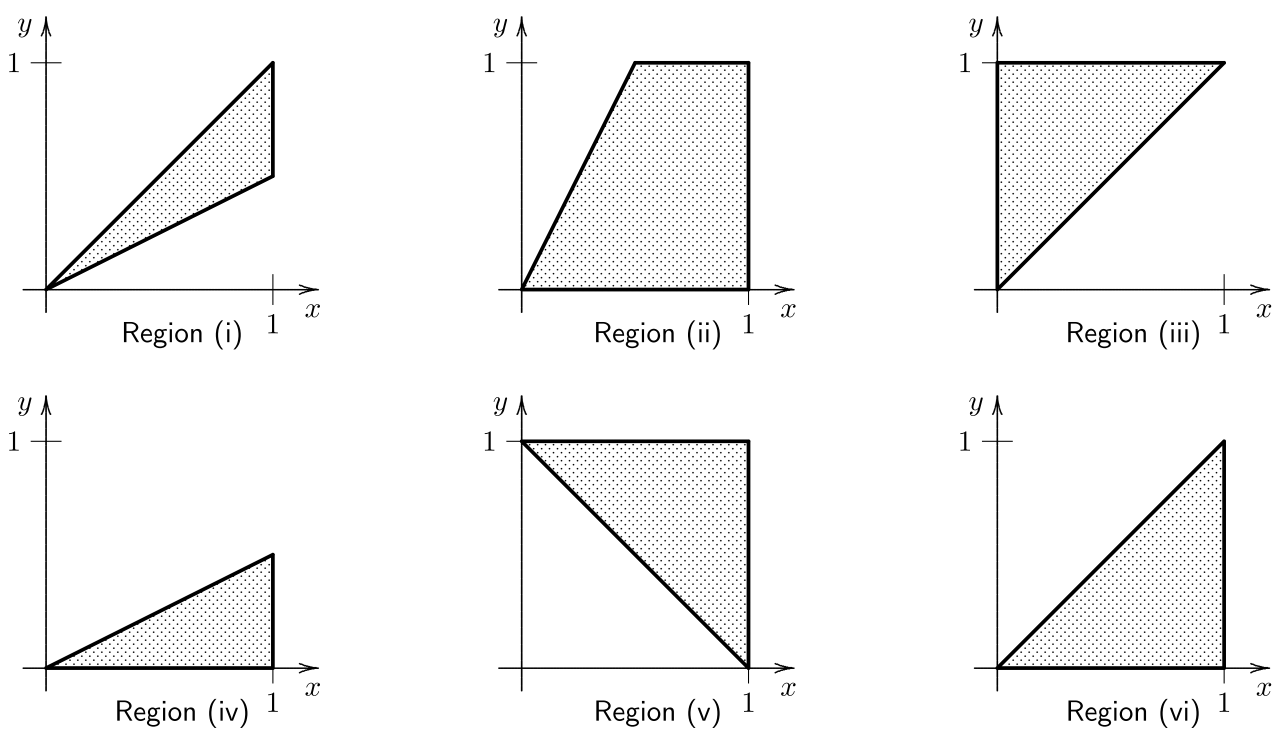


Figure 1: Figure For Problem 5.

5 Let $z = f(x, y)$ have the contour plot shown in Figure 1 below, where the horizontal axis is the x-axis and the vertical axis is the y-axis. At which of the labelled points below is the directional derivative in the direction $\mathbf{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ the greatest?

(a) A (b) B (c) C (d) D

6 Match the regions with the corresponding double integrals:



(a) $\int_0^1 \int_{x/2}^x f(x,y)\,dy\,dx$ (b) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$ (c) $\int_0^1 \int_0^{x/2} f(x,y)\,dy\,dx$

(d) $\int_0^1 \int_{y/2}^1 f(x,y)\,dx\,dy$ (e) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$ (f) $\int_0^1 \int_{1-x}^1 f(x,y)\,dy\,dx$

7 Answer the following questions True or False:

(a) **T F** The directional derivative $D_{\mathbf{v}}f$ is a vector perpendicular to $\mathbf{v}$.

(b) **T F** Given a curve $\mathbf{r}(t)$ on a surface $g(x,y,z)=1$, then $\frac{d}{dt}g(\mathbf{r}(t))=0$.

(c) **T F** If $f(x,y)$ has a local maximum at $(0,0)$, then it is possible that $f_{xx}(0,0)>0$ and $f_{yy}(0,0)<0$.

(d) **T F** Fubini's theorem assures us that $\int_0^1 \int_0^x f(x,y)\,dy\,dx = \int_0^1 \int_0^y f(x,y)\,dx\,dy$.

(e) **T F** If $x+\sin(xy)=1$, then $\frac{dy}{dx} = -\frac{1+y\cos(xy)}{x\cos(xy)}$.

(f) **T F** The directional derivative $D_{\mathbf{v}}f(1,1)$ is zero if $\mathbf{v}$ is a unit vector tangent to the level curve of f which passes through the point $(1,1)$.

PART II: Free response questions. You should attempt all parts of each problem. Show your work!

8 Let $f(x,y)=y^2-x^2$.

(a) Calculate the gradient of f, ∇f.

(b) What is the directional derivative of f in the direction $\mathbf{v}=-3\,\mathbf{i}+4\,\mathbf{j}$ at the point $(1,1)$?

(c) Find the maximum and minimum values of $f(x,y)=y^2-x^2$ subject to the constraint $g(x,y)=x^2+4y^2=4$.

9 (a) Locate and classify all the critical points of

$$f(x,y)=3y-y^3-3x^2y.$$

(b) Where on the parameterized surface

$$\mathbf{r}(x,y)=\langle u,v,w\rangle = \left\langle xy^3, \frac{x^2}{2}, \frac{3y^2}{2}\right\rangle$$

is the function $g(u,v,w)=u-v-w$ extremal? To investigate this, find all the critical points of the function $f(x,y)=xy^3-\frac{x^2}{2}-\frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum, or a saddle point and show how you know.

10 Find an equation of the tangent plane to the surface $x^2y+e^{xz}+yz=3$ at the point $(0,1,2)$.

11 Evaluate the double integral $\int_0^4 \int_0^{y^2} \frac{x^4}{4-\sqrt{x}}\,dx\,dy$.

12 Let $g(x,y,z)=x^2+2y^2-z-3$.

(a) Find the equation of the tangent plane to the level surface $g(x,y,z)=0$ at the point $(x_0,y_0,z_0)=(2,0,1)$.

(b) The surface in part (a) is the graph $z=f(x,y)$ of a function of two variables. Find the tangent line to the level curve $f(x,y)=1$ at the point $(x_0,y_0)=(2,0)$.

13 (a) Use the technique of linear approximation to estimate $f(\frac{\pi}{2}+0.1, 2.9)$ for

$$f(x,y)=\left(10\sin(x)-5y^2+8\right)^{1/3}.$$

(b) Find the unit vector at $(\frac{\pi}{2},3)$ in the direction where this function increases fastest.

14 A solid cone of height h and with base radius r has volume $f(h,r)=\frac{1}{3}\pi hr^2$ and surface area $g(h,r)=\pi r\sqrt{r^2+h^2}+\pi r^2$. Among all cones with fixed surface area $g(h,r)=\pi$, find the cone with maximal volume using the method of Lagrange multipliers.