SECOND OLD EXAM TWO SOLUTIONS

PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

1 If the function $y = f(x)$ satisfies the relation $x^2y - x + \sin y = 3$, then what is $\frac{dy}{dx}$ (in terms of x and y)?

(a) $\dfrac{1-2xy}{x^2+\sin y}$; (b) $\dfrac{1-2xy}{x^2+\cos y}$; (c) $\dfrac{1}{x^2+\sin y}$

(d) $\dfrac{1}{x^2+\cos y}$ (e) $\dfrac{4-2xy}{x^2+\sin y}$ (f) $\dfrac{4-2xy}{x^2+\cos y}$

(g) $\dfrac{1}{2x+\sin y}$ (h) $\dfrac{1}{2x+\cos y}$ (i) $\dfrac{1}{2x}$

(j) $\dfrac{4}{2x+\cos y}$

Solution: The answer is (b): $\dfrac{dy}{dx} = \dfrac{1-2xy}{x^2+\cos(y)}$.

2 The directional derivative $D_{\mathbf{u}}f(1,0)$ of the function $f(x,y) = xe^{-xy}$ at the point $(1,0)$, in the direction of the unit vector $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ equals...

(a) 0 (b) 1 (c) $\dfrac{1}{5}$ (d) $-\dfrac{1}{5}$

(e) $\dfrac{3}{5}$ (f) $-\dfrac{3}{5}$ (g) $\dfrac{4}{5}$ (h) $-\dfrac{4}{5}$

(i) $\dfrac{7}{5}$ (j) $-\dfrac{7}{5}$

Solution: The correct answer is (d):

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle e^{-xy} - xye^{-xy}, -x^2e^{-xy} \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$

and so

$$D_{\mathbf{u}}f(1,0) = \langle e^0 - (1)(0)e^0, -(1)^2e^0 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle = \langle 1, -1 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle = -\frac{1}{5}$$

3 On the closed rectangular region with vertices $(0,0)$, $(2,0)$, $(0,2)$, $(2,2)$, the function $f(x,y) = x^2 - 2xy + \frac{1}{2}y^2 + 2x$ has (absolute) extrema as follows:

(a) both the maximum and the minimum occur at interior points
(b) the maximum occurs at an interior point, the minimum on the boundary
(c) the minimum occurs at an interior point, the maximum on the boundary
(d) both the maximum and the minimum occur at boundary points
(e) the function fails to have a global maximum and/or a global minimum

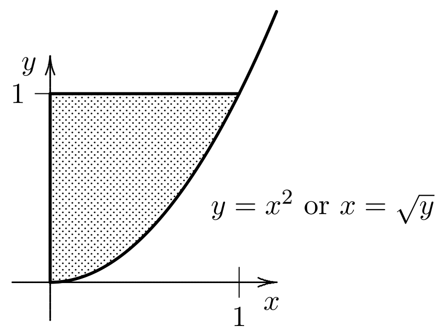
Solution: The correct answer is (d). The only way for the absolute extrema (maximum and minimum) to be interior points is for these points to be critical points (where $\nabla f = \mathbf{0}$). But $\nabla f = \langle 2x - 2y + 2, -2x + y \rangle$. We solve the two equations $2x - 2y + 2 = 0$ and $-2x + y = 0$ to find that they have solution $(x,y) = (1,2)$; thus the only critical point is a point that lies on the boundary of our rectangular region! Thus there is no extremal point in the interior, so both the maximum and the minimum occur at boundary points.

4 What is the value of the integral $\int_0^1 \int_{x^2}^1 x e^{y^2}\,dy\,dx$?
Hint: it may help to reverse the order of integration)

(a) $\frac{1}{4}(e-1)$ (b) $\frac{1}{2}(e-1)$ (c) $e-1$

(d) $2(e-1)$ (e) $4(e-1)$ (f) none of the above

Solution: Following the hint, we switch the order of integration. It helps to draw a picture of the region of integration:



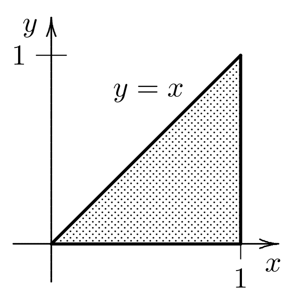
Now we switch the order and integrate:

$$\begin{aligned}\int_0^1 \int_{x^2}^1 x e^{y^2}\,dy\,dx &= \int_0^1 \int_0^{\sqrt{y}} x e^{y^2}\,dx\,dy \\ &= \int_0^1 \frac{1}{2} x^2 e^{y^2}\Big|_0^{\sqrt{y}}\,dy = \int_0^1 \frac{1}{2} y e^{y^2}\,dy \\ &= \frac{1}{4} e^{y^2}\Big|_0^1 = \frac{1}{4}(e-1).\end{aligned}$$

5 Match the integrals with those obtained by changing the order of integration. No justifications are required, but notice that one of the integrals I–V will not be used.

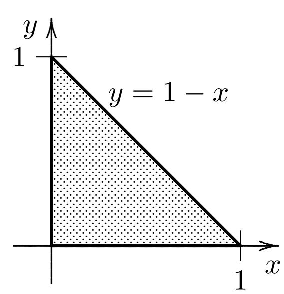
(I) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$ is the same as (b) $\int_0^1 \int_y^1 f(x,y)\,dx\,dy$

Here's the region of integration:



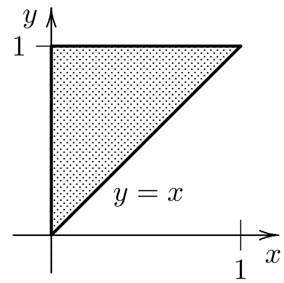
(II) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$ is the same as (c) $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$

Here's the region of integration:



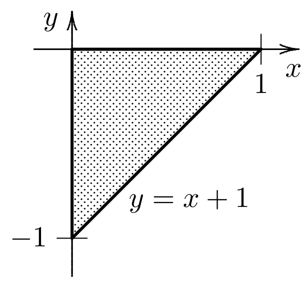
(III) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$ is the same as (d) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

Here's the region of integration:



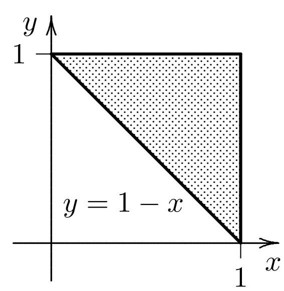
(IV) $\int_0^1 \int_0^{x-1} f(x,y)\,dy\,dx$ would be $-\int_{-1}^0 \int_0^{y+1} f(x,y)\,dx\,dy$, which isn't a choice. (This is a little confusing because the inside integral in the original expression means $0 \le y \le x-1$, but really $x-1 \le y \le 0$ in this region. That's why the negative sign appears.)

Here's the region of integration:



(V) $\int_0^1 \int_{1-x}^1 f(x,y)\,dy\,dx$ is the same as (a) $\int_0^1 \int_{1-y}^1 f(x,y)\,dx\,dy$

Here's the region of integration:



6 Answer the following questions True or False:

(a) **T F** If a function $f(x,y) = ax + by$ has a critical point, then $f(x,y) = 0$ for all (x,y).

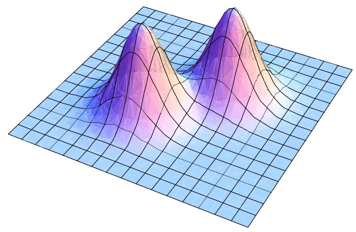
Solution: This is **True**. Remember that a critical point is where $\nabla f = \langle 0, 0\rangle$. Here $\nabla f = \langle a, b\rangle$, so to have a critical point we must have $a = b = 0$. This means $f(x,y) = 0$ for all (x,y).

(b) **T F** If (x_0, y_0) is the maximum of $f(x,y)$ on the disc $x^2 + y^2 \le 1$, then $x_0^2 + y_0^2 < 1$.

Solution: This is **False**. This statement says that if a point on the unit disk is a maximum for $f(x,y)$, then this point must be an interior point. That's just not true. (Here are some examples where the maximum occurs on the boundary: $f(x,y) = y$, $f(x,y) = x^2 + y^2$, $f(x,y) = xy$.)

(c) **T F** If $f(x,y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Solution: This is **False**. Here's a quickly sketched example:



Note that in between the two maxima is a saddle point, not a minimum.

(d) **T F** There exists a function $f(x, y)$ of two variables which has no critical points at all.

Solution: This is **True**. Go back to part (a). This function has no critical points (except when $a = b = 0$).

(e) **T F** Every critical point of a function $f(x, y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

Solution: This is **False**. It could, of course, be a saddle point if $D < 0$.

(f) **T F** If $(0, 0)$ is a critical point of a function $f(x, y)$ where the discriminant D is zero but $f_{xx}(0, 0) < 0$, then $(0, 0)$ cannot be a local minimum.

Solution: This is **True**. This is saying that the trace in the xz-plane is concave down at our critical point, which means that the point cannot be a local minimum (which is concave up).

Part II: Free response questions. You should attempt all parts of each problem. Show your work!

7 Find the point on the surface $xy^2z^3 = 6\sqrt{3}$ in the first octant (that is, with $x > 0$, $y > 0$, $z > 0$) that is closest to the origin.

Solution: Here we want to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ (for simplicity, we use the square of the distance to the origin rather than the distance itself) subject to the constraint that $g(x, y, z) = xy^2z^3 = 6\sqrt{3}$. Thus we use Lagrange multipliers.

We start with the equations $\nabla f = \lambda \nabla g$. Notice that x, y, and z are all assumed to be positive (and so not zero), so we can solve for λ in each case:

$$2x = \lambda y^2 z^3 \quad \text{or} \quad \lambda = \frac{2x}{y^2 z^3} \tag{1}$$

$$2y = 2\lambda xyz^3 \quad \text{or} \quad \lambda = \frac{1}{xz^3} \tag{2}$$

$$2z = 3\lambda xy^2 z^2 \quad \text{or} \quad \lambda = \frac{2}{3xy^2 z} \tag{3}$$

Setting equal the two expressions for λ from equations (1) and (2), we get $y^2 = 2x^2$ or $y = \sqrt{2}\,x$. Similarly, equating the expressions for λ from equations (1) and (3), we get $z^2 = 3x^2$ or $z = \sqrt{3}\,x$. Plugging both these into the constraint $g(x, y, z) = 6\sqrt{3}$, we get $x(\sqrt{2}\,x)^2(\sqrt{3}\,x)^3 = 6\sqrt{3}$, or $6\sqrt{3}\,x^6 = 6\sqrt{3}$. Thus $x^6 = 1$ and so (since $x > 0$) $x = 1$. Hence our point is $(x, y, z) = (1, \sqrt{2}, \sqrt{3})$.

8 (a) Find and classify (each as a local minimum, a local maximum, or a saddle point) all the critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 4$.

Solution: Remember that critical points of $f(x, y)$ are those points where $\nabla f = \langle 0, 0\rangle$. Here $\nabla f = \langle 4x^3 - 4y, 4y^3 - 4x\rangle$, so critical points are those points where $y = x^3$ and $x = y^3$. Thus $y = (y^3)^3 = y^9$. This can also be written as $y^9 - y = 0$, which factors into $y(y^4 + 1)(y^2 + 1)(y + 1)(y - 1) = 0$, so $y = 0$, $y = -1$, or $y = 1$. Thus we get the points $(0, 0)$, $(-1, -1)$, and $(1, 1)$.

We turn to the second derivative test to classify these points. Since $f_{xx} = 12x^2$, $f_{xy} = f_{yx} = -4$, and $f_{yy} = 12y^2$, we get $D = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$. Thus we can classify these points as follows:

Critical Point	$(0,0)$	$(-1,-1)$	$(1,1)$
Value of $f(x,y)$	4	2	2
D	-16	128	128
f_{xx}	0	12	12
Classification	Saddle	Local Min	Local Min

(b) Does the function $f(x,y)$ of part (a) have a global maximum? If yes, at which point(s) is the global maximum attained? If no, why?

Solution: This function has no global maximum. If we look at the points where $y = 0$, our function is simply $f(x,0) = x^4 + 4$. As x grows without bound, so does $f(x,y)$. Thus there can be no global maximum.

(c) Does the function $f(x,y)$ of part (a) have a global minimum? If yes, at which point(s) is the global minimum attained? If no, why?

Solution: The global minimum is $f(x,y) = 2$, which is attained at the points $(x,y) = (1,1)$ and $(x,y) = (-1,-1)$ (critical points from part (a)). How can we tell that these are global minima and not simply local minima?

One way to do this is to restrict ourselves to a giant disk $x^2 + y^2 \leq a^2$. We know the only critical points inside the disk are those found in part (a). We can then use Lagrange Multipliers to find that the extremal points on the boundary occur at one of six points: where $x = 0$ (so $(0, \pm a)$), where $y = 0$ (so $(\pm a, 0)$) or where $xy = -1$. In all these cases we get a large value of $f(x,y)$: either $a^4 + 4$ (in the cases where $x = 0$ or $y = 0$) or $x^4 + y^4 + 8 \geq 8$ (when $xy = -1$). Thus all these points are greater than the value $f(x,y) = 2$ which occurs at our local minima.

Since our minimum on every giant disk is $f(1,1) = f(-1,-1) = 2$, our absolute minimum on the entire plane must also be 2.

9 What point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ is closest to the origin?

Solution: This is very similar to Problem 7, above. Again we minimize $f(x,y,z) = x^2 + y^2 + z^2$ (for ease of computation we're using the square of the distance to the origin rather than the actual distance) subject to the constraint $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$. We use Lagrange multipliers and solve the equations $\nabla f = \lambda \nabla g$ together with $g(x,y,z) = 1$. These equations are

$$2x = \lambda\left(-\frac{1}{x^2}\right)$$

$$2y = \lambda\left(-\frac{1}{y^2}\right)$$

$$2z = \lambda\left(-\frac{8}{z^2}\right)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1.$$

The first three equations give us three expressions for λ, namely $\lambda = -2x^3 = -2y^3 = -\frac{1}{4}z^3$. Thus $y = x$ and $z = 2x$, and so plugging in to the last equation $\frac{1}{x} + \frac{1}{x} + \frac{8}{2x} = 1$ implies $\frac{6}{x} = 1$, or $x = 6$. Thus $(x,y,z) = (6,6,12)$ is the point closest to the origin.

Or so it would appear. An interesting twist was discovered in a previous semester by a student (Jacob Aptekar). The point $(x,y,z) = (1, -\frac{1}{n}, -\frac{8}{n})$ lies on this surface $g(x,y,z) = 1$. But as n grows without bound, this point approaches $(1,0,0)$. In particular, the distance to the origin of these points approaches 1. So the point above is a *local* minimum but not a *global* minimum, and in particular is *not* the point on the surface closest to the origin.

10 Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find an absolute maximum or absolute minimum among them?

Solution: The critical points are where $\nabla f = \mathbf{0}$, or $\langle 3x^2 - 3, 3y^2 - 12\rangle = \langle 0, 0\rangle$. Thus $x = \pm 1$ and $y = \pm 2$, so we have four critical points: $(\pm 1, \pm 2)$. Since $f_{xx} = 6x$, $f_{xy} = f_{yx} = 0$, and $f_{yy} = 6y$, our discriminant $D = 36xy$. Thus we classify our critical points in the usual way:

Critical Point	$(1,2)$	$(-1,2)$	$(1,-2)$	$(-1,-2)$
Value of $f(x,y)$	2	6	34	38
D	72	-72	-72	72
f_{xx}	6	-6	6	-6
Classification	Local Min	Saddle	Saddle	Local Max

Are either of our local extrema actually global extrema? If we restrict to $y = 0$, for example, we get $f(x, 0) = x^3 - 3x + 20$. This function grows without bound as $x \to \infty$ and it grows negatively without bound as $x \to -\infty$. Thus there can be no global maximum or minimum.

11 Consider the equation

$$f(x, y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0.$$

This defines a curve that passes through the point $(1, 1)$. Near this point, the curve can be written as a graph $y = g(x)$. Find the slope of that graph at the point $(1, 1)$.

Solution: We use the chain rule to perform the implicit differentiation. That is, $f_x \cdot 1 + f_y \cdot \frac{dy}{dx} = 0$, so $(2xy^2 - 4y + 4x^3) + (6y^2 + 2x^2y - 4x)\frac{dy}{dx} = 0$. At $(x, y) = (1, 1)$, this means that $2 + 4\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{1}{2}$. This is the slope requested.

Another approach is to use the gradient $\nabla f = \langle 2, 4\rangle$ (at $(1, 1)$) to find the equation of the tangent line at $(1, 1)$:

$$\langle 2, 4\rangle \cdot \langle x - 1, y - 1\rangle = 0 \qquad \text{or} \qquad 2x + 4y = 6 \qquad \text{or} \qquad y = -\frac{1}{2}x + \frac{3}{2}.$$

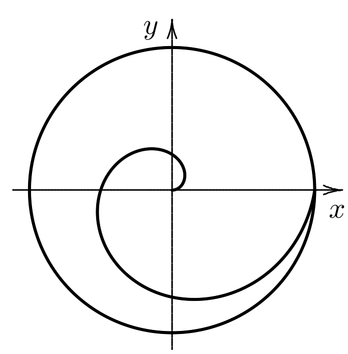
This line also has slope $-\frac{1}{2}$.

12 Evaluate the double integral

$$\iint\limits_R \sqrt{x^2 + y^2}\, dx\, dy$$

where R is the region bounded by the positive x-axis, the spiral curve $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$ $(0 \le t \le 2\pi)$ and the circle with radius 2π (centered at the origin).

Solution: Here's a quick sketch of the region of integration:



Clearly we'll prefer polar coordinates to rectangular coordinates. In polar coordinates, the region can be described as $\theta \le r \le 2\pi$ with $0 \le \theta \le 2\pi$ and the integrand is $\sqrt{x^2 + y^2} = r$. Thus, replacing

$dx\ dy$ with $r\ dr\ d\theta$, we have

$$\iint_R \sqrt{x^2+y^2}\ dx\ dy = \int_0^{2\pi}\int_\theta^{2\pi} r\ r\ dr\ d\theta = \int_0^{2\pi}\int_\theta^{2\pi} r^2\ dr\ d\theta$$
$$= \int_0^{2\pi} \frac{1}{3}r^3\Big|_\theta^{2\pi}\ d\theta = \frac{1}{3}\int_0^{2\pi}\left(8\pi^3-\theta^3\right)\ d\theta$$
$$= \frac{1}{3}\left[8\pi^3\theta - \frac{1}{4}\theta^4\right]_0^{2\pi} = \frac{1}{3}\left(16\pi^4\theta - 4\pi^4\right) = 4\pi^4.$$

13 (a) Integrate $f(x,y) = x^2-y^2$ over the unit disk $\{x^2+y^2\le 1\}$.

Solution: This region of integration calls out for polar coordinates, so we replace x^2-y^2 with $r^2\left(\cos^2(\theta)-\sin^2(\theta)\right) = r^2\cos(2\theta)$. Thus the integral is

$$\int_0^{2\pi}\int_0^1 r^2\cos(2\theta)\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 r^3\cos(2\theta)\ dr\ d\theta$$
$$= \int_0^{2\pi}\frac{1}{4}r^4\cos(2\theta)\Big|_0^1\ d\theta = \frac{1}{4}\int_0^{2\pi}\cos(2\theta)\ d\theta$$
$$= \frac{1}{8}\sin(2\theta)\Big|_0^{2\pi} = \frac{1}{8}\left(\sin(4\pi)-\sin(0)\right) = 0.$$

Thus the integral is zero.

(b) Here is a challenging integral:

$$\int_0^1\int_0^{\sqrt{1-\theta^2}} r^2\ dr\ d\theta.$$

Hint: Does it matter that the variables are named r and θ? Could they have been x and y?

Solution: Here the trick is to consider the integral

$$\int_0^1\int_0^{\sqrt{1-x^2}} y^2\ dy\ dx. \qquad (*)$$

This is the exact same integral (well, r has been replaced by y and θ by x, but it has the same value, right?). So let's focus on this new integral.

This new integral is over the part of the unit disk in the first quadrant, so it's natural to use polar coordinates. We do the usual substitutions ($y = r\sin(\theta)$, $dy\ dx = r\ dr\ d\theta$) to get

$$\int_0^1\int_0^{\sqrt{1-x^2}} y^2\ dy\ dx = \int_0^{\pi/2}\int_0^1 r^2\sin^2(\theta)\ r\ dr\ d\theta = \int_0^{\pi/2}\int_0^1 r^3\sin^2(\theta)\ dr\ d\theta$$
$$= \int_0^{\pi/2}\frac{1}{4}r^4\sin^2(\theta)\Big|_{r=0}^1\ d\theta = \frac{1}{4}\int_0^{\pi/2}\sin^2(\theta)\ d\theta$$
$$= \frac{1}{8}\int_0^{\pi/2}\left(1-\cos(2\theta)\right)\ d\theta = \frac{1}{8}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2}$$
$$= \frac{1}{8}\left(\frac{\pi}{2} - \frac{1}{2}\big(\sin(\pi)-\sin(0)\big)\right) = \frac{1}{8}\left(\frac{\pi}{2}-\frac{1}{2}(0-0)\right) = \frac{\pi}{16}.$$

Notice that we've used the double-angle formula $\sin^2(\theta) = \frac{1}{2}\left(1-\cos(2\theta)\right)$.

The real confusion in this problem comes from not starting with the integral $(*)$. Since our original integral had r and θ as the variables, when we eventually switch to polar coordinates it feels unfair.