SECOND OLD EXAM TWO

PART I: Multiple choice. Each problem has a unique correct answer. You do not need to justify your answers in this part of the exam.

1 If the function $y = f(x)$ satisfies the relation $x^2y - x + \sin y = 3$, then what is $\frac{dy}{dx}$ (in terms of x and y)?

(a) $\dfrac{1-2xy}{x^2+\sin y}$; (b) $\dfrac{1-2xy}{x^2+\cos y}$; (c) $\dfrac{1}{x^2+\sin y}$

(d) $\dfrac{1}{x^2+\cos y}$ (e) $\dfrac{4-2xy}{x^2+\sin y}$ (f) $\dfrac{4-2xy}{x^2+\cos y}$

(g) $\dfrac{1}{2x+\sin y}$ (h) $\dfrac{1}{2x+\cos y}$ (i) $\dfrac{1}{2x}$

(j) $\dfrac{4}{2x+\cos y}$

2 The directional derivative $D_{\mathbf{u}}f(1,0)$ of the function $f(x,y) = xe^{-xy}$ at the point $(1,0)$, in the direction of the unit vector $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ equals...

(a) 0 (b) 1 (c) $\dfrac{1}{5}$ (d) $-\dfrac{1}{5}$

(e) $\dfrac{3}{5}$ (f) $-\dfrac{3}{5}$ (g) $\dfrac{4}{5}$ (h) $-\dfrac{4}{5}$

(i) $\dfrac{7}{5}$ (j) $-\dfrac{7}{5}$

3 On the closed rectangular region with vertices $(0,0)$, $(2,0)$, $(0,2)$, $(2,2)$, the function $f(x,y) = x^2 - 2xy + \frac{1}{2}y^2 + 2x$ has (absolute) extrema as follows:

(a) both the maximum and the minimum occur at interior points
(b) the maximum occurs at an interior point, the minimum on the boundary
(c) the minimum occurs at an interior point, the maximum on the boundary
(d) both the maximum and the minimum occur at boundary points
(e) the function fails to have a global maximum and/or a global minimum

4 What is the value of the integral $\displaystyle\int_0^1 \int_{x^2}^1 xe^{y^2}\, dy\, dx$?
Hint: it may help to reverse the order of integration)

(a) $\dfrac{1}{4}(e-1)$ (b) $\dfrac{1}{2}(e-1)$ (c) $e-1$

(d) $2(e-1)$ (e) $4(e-1)$ (f) none of the above

5 Match the integrals with those obtained by changing the order of integration. No justifications are required, but notice that one of the integrals I–V will not be used.

(I) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

(II) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$

(III) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

(IV) $\int_0^1 \int_0^{x-1} f(x,y)\,dy\,dx$

(V) $\int_0^1 \int_{1-x}^1 f(x,y)\,dy\,dx$

(a) $\int_0^1 \int_{1-y}^1 f(x,y)\,dx\,dy$

(b) $\int_0^1 \int_y^1 f(x,y)\,dx\,dy$

(c) $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$

(d) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

6 Answer the following questions True or False:

(a) **T F** If a function $f(x,y) = ax + by$ has a critical point, then $f(x,y) = 0$ for all (x,y).

(b) **T F** If (x_0, y_0) is the maximum of $f(x,y)$ on the disc $x^2 + y^2 \le 1$, then $x_0^2 + y_0^2 < 1$.

(c) **T F** If $f(x,y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

(d) **T F** There exists a function $f(x,y)$ of two variables which has no critical points at all.

(e) **T F** Every critical point of a function $f(x,y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

(f) **T F** If $(0,0)$ is a critical point of a function $f(x,y)$ where the discriminant D is zero but $f_{xx}(0,0) < 0$, then $(0,0)$ cannot be a local minimum.

PART II: Free response questions. You should attempt all parts of each problem. Show your work!

7 Find the point on the surface $xy^2z^3 = 6\sqrt{3}$ in the first octant (that is, with $x > 0$, $y > 0$, $z > 0$) that is closest to the origin.

8 (a) Find and classify (each as a local minimum, a local maximum, or a saddle point) all the critical points of the function $f(x,y) = x^4 + y^4 - 4xy + 4$.

(b) Does the function $f(x,y)$ of part (a) have a global maximum? If yes, at which point(s) is the global maximum attained? If no, why?

(c) Does the function $f(x,y)$ of part (a) have a global minimum? If yes, at which point(s) is the global minimum attained? If no, why?

9 What point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ is closest to the origin?

10 Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find an absolute maximum or absolute minimum among them?

11 Consider the equation

$$f(x,y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0.$$

This defines a curve that passes through the point $(1,1)$. Near this point, the curve can be written as a graph $y = g(x)$. Find the slope of that graph at the point $(1,1)$.

12 Evaluate the double integral

$$\iint_R \sqrt{x^2+y^2}\, dx\, dy$$

where R is the region bounded by the positive x-axis, the spiral curve $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$ $(0 \le t \le 2\pi)$ and the circle with radius 2π (centered at the origin).

13 (a) Integrate $f(x,y) = x^2 - y^2$ over the unit disk $\{x^2+y^2 \le 1\}$.

(b) Here is a challenging integral:

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2\, dr\, d\theta.$$

Hint: Does it matter that the variables are named r and θ? Could they have been x and y?