1 Suppose $F(x,y) = 2e^{x-y} + 2$.

(a) Write down an equation for the tangent plane to the graph of $F(x,y)$ at the point where $x = y = 10$.

Solution: The tangent plane has equation

$$z - z_0 = F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0),$$

where $(x_0,y_0) = (10,10)$ and $z_0 = F(x_0,y_0) = 2e^{10-10} + 2 = 4$. We compute the derivatives to find $F_x = 2e^{x-y}$ and $F_y = -2e^{x-y}$, so $F_x(10,10) = 2$ while $F_y(10,10) = -2$. Thus the tangent plane is

$$z - 4 = 2(x-10) - 2(y-10) \quad \text{or} \quad z = 2x - 2y + 4.$$

(b) Estimate the value of $F(10.1, 10.2)$ to one decimal place using the technique of linear approximation.

Solution: This is simply just the approximation of $F(x,y)$ by the tangent plane. That is, to approximate $F(x,y)$, we use

$$F(x,y) \approx F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0),$$

which is exactly the equation from part (a). Thus for (x,y) near $(10,10)$,

$$F(x,y) \approx 4 + 2(x-10) - 2(y-10).$$

In particular,

$$F(10.1, 10.2) \approx 4 + 2(10.1-10) - 2(10.2-10) = 3.8.$$

2 Consider the function $F(x,y) = x^2 + y^2 + x^2 y$.

(a) Find all the critical points of the function $F(x,y)$.

Solution: The critical points of $F(x,y)$ are those points where $\nabla F = \mathbf{0}$ or $\langle F_x, F_y\rangle = \langle 0,0\rangle$. Here $\nabla F = \langle 2x+2xy, 2y+x^2\rangle$, so we're solving the equations $2x(1+y) = 0$ and $2y = -x^2$. The first equation implies $x = 0$ or $y = -1$. If $x = 0$, then the second equation implies $y = 0$ as well. If $y = -1$, then $x = \pm\sqrt{2}$, so we have three critical points: $(0,0)$, $(\sqrt{2},-1)$, and $(-\sqrt{2},-1)$.

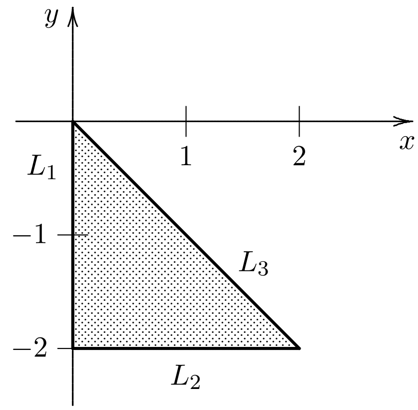
(b) For each of the critical points you found in part (a), decide whether it is a local maximum, local minimum, or a saddle for $F(x,y)$.

Solution: Here $D = F_{xx}F_{yy} - F_{xy}^2 = (2+2y)(2) - (2x)^2$, so we get the following classification:

Critical Point	$(0,0)$	$(\sqrt{2},-1)$	$(-\sqrt{2},-1)$
Value of $F(x,y)$	0	1	1
D	4	-8	-8
f_{xx}	2	0	0
Classification	Local Min	Saddle	Saddle

(c) Find the maximum value of $F(x,y)$ on the triangular region with vertices $(0,0)$, $(0,-2)$, and $(2,-2)$. The region includes the boundary of the triangle.

Solution: Here is a picture of the region, with the boundary segments labeled L_1, L_2, and L_3:



There are no critical points inside this region, so the maximum must occur on the boundary. We consider each piece in turn:

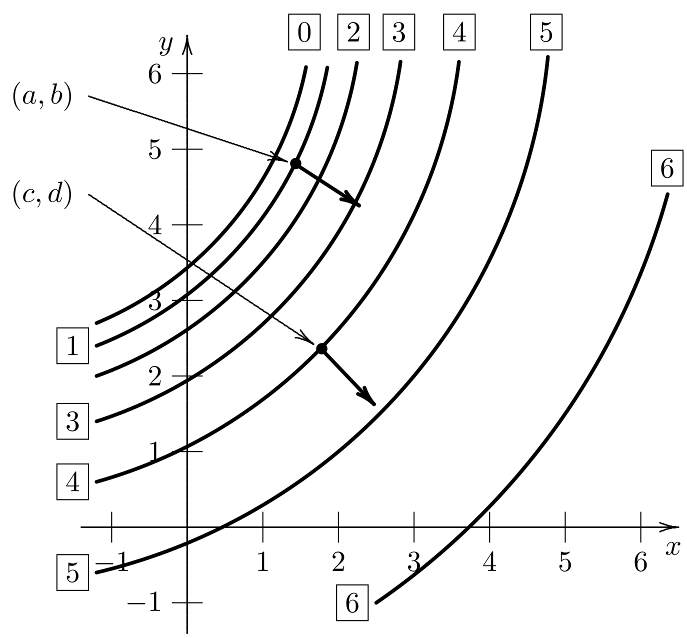
On L_1, we have $x = 0$ and so $F(0, y) = y^2$. This function varies from $F(0, 0) = 0$ to $F(0, -2) = 4$. Thus the maximum on L_1 is $F(0, -2) = 4$.

On L_2, we have $y = -2$ and so $F(x, -2) = 4 - x^2$. This function varies from $F(0, -2) = 4$ to $F(2, -2) = 0$. Thus the maximum on L_2 is $F(0, -2) = 4$.

On L_3, we have $y = -x$ and so $F(x, -x) = 2x^2 - x^3 = x^2(2 - x)$. If we think of this as $g(x)$, we can find that $g'(x) = 4x - 3x^2 = x(4 - 3x)$, so this has a critical point at $x = \frac{4}{3}$. Since $g''(x) = 4 - 6x$ is negative at this point, we see that this is a maximum. Thus on L_3 the maximum is $F(\frac{4}{3}, -\frac{4}{3}) = \frac{32}{27} \approx 1.185$.

Thus the largest value of $F(x, y)$ on the boundary (and thus on the entire triangular region) is $F(0, -2) = 4$.

3 The contour plot of the function $f(x, y)$ is given in the figure below.



(a) Indicate whether the statements below are true or false, and give an explanation.

(i) $f_x(a, b) \neq 0$

Solution: This is **True**. As we travel from in the x-direction (from left to right on a horizontal line) through (a, b), the function $f(x, y)$ is increasing. We see this as we're passing from the $f = 0$, through the $f = 1$ curve, and toward the $f = 2$ curve.

(ii) $f_x(a, b) > f_x(c, d)$

Solution: This is **True**. This is saying that the contours are closer together on horizontal lines through (a, b) than they are on horizontal lines through (c, d), which is true.

(iii) $f_x(a,b) < f_y(a,b)$

Solution: This is **False**. In fact, $f_y(a,b)$ is negative, as the contours are decreasing as we move vertically (in the y-direction).

(iv) $f_{xx}(a,b) > 0$

Solution: This is **False**. As we move horizontally at (a,b), the contours are becoming more spaced apart. This means the rate of change f_x is becoming smaller, so $f_{xx}(a,b) < 0$.

(b) At each of the labeled points, draw a unit vector in the direction of ∇f at the point.

Solution: The unit vectors are sketched on the graph. The key facts are these:

- The gradient vectors are perpendicular to the tangent lines of the level curves, and
- The gradient vectors point in the direction of increasing contours.

Keep in mind that the vectors are only sketches.

(c) Which vector is longer, $\nabla f(a,b)$ or $\nabla f(c,d)$?

Solution: Recall that $\nabla f = \langle f_x, f_y\rangle$ represents the "total" rate of change of $f(x,y)$. At (a,b) the function is increasing quickly – the contours are very close together – while by (c,d) the function is still increasing, but at a more gradual pace. Thus $\nabla f(a,b)$ should have a greater magnitude (be a longer vector) than $\nabla f(c,d)$.

4 A moth's position at time t seconds is given by the position vector $\mathbf{r}(t) = \langle \cos(\pi t), t\sin(\pi t), 10\rangle$ (for $t \geq 0$). Suppose the temperature at any point in space is given in degrees Fahrenheit by $T(x,y,z) = 2xz + y^2 + z^3 + 40$. What is the rate of change of the temperature (in degrees per second) as experienced by the moth at time $t = 1$?

Solution: At $t = 1$ the moth is at position $\mathbf{r}(1) = \langle \cos(\pi), \sin(\pi), 10\rangle = \langle -1, 0, 10\rangle$. We could find the function T explicitly as a function of time t, but we'll avoid this Calculus 1 solution in favor of using the multivariable chain rule:

$$\frac{d}{dt}T(\mathbf{r}(t)) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

We compute the gradient $\nabla T = \left\langle 2z, 2y, 2x + 3z^2\right\rangle$. When $t = 1$, we've seen that $\langle x,y,z\rangle = \langle -1, 0, 10\rangle$, so $\nabla T = \langle 20, 0, 298\rangle$. On the other hand, $\mathbf{r}'(t) = \langle -\pi\sin(\pi t), \sin(\pi t) + \pi t\cos(\pi t), 0\rangle$, which at $t = 1$ is $\mathbf{r}'(1) = \langle 0, -\pi, 0\rangle$. Thus, at $t = 1$,

$$\left.\frac{dT}{dt}\right|_{t=1} = \nabla T(\mathbf{r}(1)) \cdot \mathbf{r}'(1) = \langle 20, 0, 298\rangle \cdot \langle 0, -\pi, 0\rangle = 0.$$

Thus the rate of change of the temperature is zero.

Note: There was a typo in this problem that we have not corrected. Originally the position vector of the moth was $\mathbf{r}(t) = \langle t\cos(\pi t), \ldots$ rather than $\mathbf{r}(t) = \langle \cos(\pi t), \ldots$ (note the omitted t). If we used the intended position vector, we would end up with $\mathbf{r}'(1) = \langle -1, -\pi, 0\rangle$, and so

$$\left.\frac{dT}{dt}\right|_{t=1} = \nabla T(\mathbf{r}(1)) \cdot \mathbf{r}'(1) = \langle 20, 0, 298\rangle \cdot \langle -1, -\pi, 0\rangle = -20.$$

5 Let $F(x,y) = g(x^2 y)$ where g is a continuous function of one variable with continuous first and second derivatives. Calculate $F_{xy}(2,2) + F_{yx}(2,2)$ if you also know that $g'(8) = 3$ and $g''(8) = 1$.

Solution: This is another chain rule problem. When we talk about $g'(8)$, we're thinking about g as a function of one variable. We'll say $g(u)$ (g is a function of u), so $g'(8)$ just means $g'(u)$ when $u = 8$. Now we're defining $F(x, y)$ to be $g(u(x, y))$, where $u(x, y) = x^2y$. The chain rule says that

$$\frac{\partial F}{\partial x} = \frac{dg}{du}\frac{\partial u}{\partial x} = g'(u)2xy \qquad \text{and} \qquad \frac{\partial F}{\partial y} = \frac{dg}{du}\frac{\partial u}{\partial y} = g'(u)x^2.$$

We're actually interested in F_{xy}, which is the derivative of F_x with respect to y. We compute this from the above using the product rule as well as the chain rule:

$$F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}\left(g'(u)2xy\right) = \left(g''(u)\frac{\partial u}{\partial y}\right)2xy + g'(u)2x$$
$$= g''(u)x^2 \cdot 2xy + g'(u)2x.$$

Now when $(x, y) = (2, 2)$, we get $u = x^2y = 8$ and so

$$F_{xy}(2, 2) = g''(8)(2)^2 \cdot 2(2)(2) + g'(8)2(2) = 32g''(8) + 4g'(8) = 32(1) + 4(3) = 44.$$

By Clairaut's theorem, $F_{yx} = F_{xy}$. This means we don't need to compute the other derivative, and so $F_{xy}(2, 2) + F_{yx}(2, 2) = 44 + 44 = 88$.

6 Suppose $F(x, y)$ is a function such that $D_{\mathbf{u}}F(1, 1) = 5$ and $D_{\mathbf{v}}F(1, 1) = \sqrt{2}$, where $\mathbf{u} = \langle 1, 0\rangle$ and $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.

(a) Find the vector $\nabla F(1, 1)$.

Solution: Recall that $D_{\mathbf{u}}F(1, 1) = \nabla F(1, 1) \cdot \mathbf{u}$. If we write $\nabla F(1, 1) = \langle a, b\rangle$, then we're told that $\langle a, b\rangle \cdot \langle 1, 0\rangle = 5$ and $\langle a, b\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \sqrt{2}$. From the first we get that $a = 5$, and from the second we get that $5 + b = 2$ or $b = -3$. Thus $\nabla F(1, 1) = \langle 5, -3\rangle$.

(b) What is the maximum possible value of $D_{\mathbf{w}}F(1, 1)$ for a unit vector $\mathbf{w}$?

Solution: Remember that $D_{\mathbf{w}}F(1, 1) = \nabla F(1, 1) \cdot \mathbf{w} = |\nabla F(1, 1)|\,|\mathbf{w}|\,\cos(\theta)$, where θ is the angle between $\nabla F(1, 1)$ and $\mathbf{w}$. Here $|\mathbf{w}| = 1$ ($\mathbf{w}$ is a unit vector) and $\cos(\theta)$ can be as large as 1. Thus the maximum possible value of $D_{\mathbf{w}}F(1, 1)$ is $|\nabla F(1, 1)|$, which is $|\langle 5, -3\rangle| = \sqrt{34}$.

(c) Find a unit vector $\mathbf{w}$ such that $D_{\mathbf{w}}F(1, 1) = 0$.

Solution: Since $D_{\mathbf{w}}F(1, 1) = \nabla F(1, 1) \cdot \mathbf{w}$, what we're really looking for is a vector perpendicular to $\nabla F(1, 1) = \langle 5, -3\rangle$. The usual trick is to reverse the components and change one sign, so a natural choice would be $\langle 3, 5\rangle$. (We can check: $\langle 5, -3\rangle \cdot \langle 3, 5\rangle = 0$.) This is not a unit vector, however, so we need to divide by the appropriate length to get $\mathbf{w} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right\rangle$.

7 Captain Kurt of the starship Interprize has steered his starship toward a wormhole in space. The Interprize's current location is $(0, 2, 2)$. The wormhole is quite oddly shaped, and its surface is given by the equation $-x^2 + 2y^2 + 2z^2 = 1$. Captain Kurt is really eager to get to the wormhole as soon as possible. Find a point on the wormhole's surface that is nearest to the Interprize so that Kurt knows where to send the ship. Be sure to write up your solution to this problem carefully, explaining all the steps involved in getting to your answer.

Solution: We're trying to minimize the function $f(x, y, z) = x^2 + (y - 2)^2 + (z - 2)^2$ (this is the square of the distance from (x, y, z) to $(0, 0, 2)$; we use the square rather than the distance to make our lives simpler) subject to the constraint $g(x, y, z) = -x^2 + 2y^2 + 2z^2 = 1$. This is clearly a job for Lagrange multipliers.

From $\nabla f = \lambda \nabla g$, we get three equations:

$$2x = \lambda(-2x) \quad (1)$$
$$2y - 4 = \lambda(4y) \quad (2)$$
$$2z - 4 = \lambda(4z). \quad (3)$$

From equation (1), we see that either $x = 0$ or $\lambda = -1$. If $\lambda = -1$, then from equations (2) and (3) we see that $y = z = \frac{2}{3}$. Plugging these values into $g(x, y, z) = 1$, we find that $x = \pm\frac{\sqrt{7}}{3}$. Thus we get the points $(x, y, z) = (\pm\frac{\sqrt{7}}{3}, \frac{2}{3}, \frac{2}{3})$.

If $x = 0$, then we turn to the other equations. From equations (2) and (3), we can see that $y = z = \frac{4}{2-4\lambda}$ (provided $\lambda \neq \frac{1}{2}$ – what happens if $\lambda = \frac{1}{2}$?). Plugging $y = z$ and $x = 0$ into $g(x, y, z) = 1$, we find that $(x, y, z) = (0, \frac{1}{2}, \frac{1}{2})$ or $(0, -\frac{1}{2}, -\frac{1}{2})$.

Now let's see what the values of $f(x, y, z)$ are at these four points:

$$f(\pm\tfrac{\sqrt{7}}{3}, \tfrac{2}{3}, \tfrac{2}{3}) = \left(\tfrac{\sqrt{7}}{3}\right)^2 + \left(\tfrac{2}{3} - 2\right)^2 + \left(\tfrac{2}{3} - 2\right)^2 = \frac{39}{9}$$

$$f(0, \tfrac{1}{2}, \tfrac{1}{2}) = (0)^2 + \left(\tfrac{1}{2} - 2\right)^2 + \left(\tfrac{1}{2} - 2\right)^2 = \frac{9}{2}$$

$$f(0, -\tfrac{1}{2}, -\tfrac{1}{2}) = (0)^2 + \left(-\tfrac{1}{2} - 2\right)^2 + \left(-\tfrac{1}{2} - 2\right)^2 = \frac{25}{2}.$$

Thus the minimum distance occurs at the points $(\pm\frac{\sqrt{7}}{3}, \frac{2}{3}, \frac{2}{3})$.

8 A battle to save a wildlife preserve in Alaska has led to Congress authorizing drilling for oil on the Ellipse, a park between the White House and the Washington Monument bounded by the curve $x^2 + 4y^2 = 100$. Geologists have studied the matter and report that the value V of the oil from a well drilled on the Ellipse will be given by the formula $V = 200 + 18y - x^2 - y^2$.

(a) The President asks you, the Secretary of Energy, to find the coordinates (x, y) of the most valuable drilling site, the least valuable drilling site, and the maximum and minimum values. Do so.

Solution: This is a standard optimization problem. The extreme values occur either at critical points in the interior of the Ellipse or at points found by Lagrange multipliers on the boundary of the Ellipse.

The critical points of V are those points where $\nabla V = \mathbf{0}$, or $\langle V_x, V_y\rangle = \langle 0, 0\rangle$. This is $\langle -2x, 18 - 2y\rangle = \langle 0, 0\rangle$. Hence $-2x = 0$ and $18 - 2y = 0$, so we get exactly one critical point: $(x, y) = (0, 9)$. Alas, this point is outside the Ellipse.

On the boundary we use Lagrange multipliers to extremize $V = 200 + 18y - x^2 - y^2$ subject to the constraint $g(x, y) = x^2 + 4y^2 = 100$. We solve the equations $\nabla V = \lambda\nabla g$ and $g(x, y) = 100$ as follows: $\nabla V = \lambda\nabla g$ gives us the pair of equations

$$-2x = \lambda 2x$$
$$18 - 2y = \lambda 8y.$$

From the first equation, either $x = 0$ or $\lambda = -1$. If $x = 0$, then the constraint equation implies that $y = \pm 5$. If $\lambda = -1$, then the second equation implies $y = -3$. Now using the constraint means that $x = \pm\sqrt{100 - 4(-3)^2} = \pm 8$.

Now we simply compare the values of V at the points we've found: $(0, \pm 5)$ and $(\pm 8, -3)$:

$$V(0, 5) = 265$$
$$V(0, -5) = 85$$
$$V(\pm 8, -3) = 73$$

Thus the maximum is $V(0, 5) = 265$ and the minimum is $V(\pm 8, -3) = 73$.

(b) The President has a follow-up question: is there a more valuable site if drilling was allowed outside the ellipse? Determine if there is a drilling site with even greater V and, if so, where it is located.

Solution: A reasonable place to check is $(0, 9)$, our critical point outside the Ellipse. Sure enough, it turns out that $V(0, 9) = 281$ is the global maximum. In any case, it's certainly more valuable than anywhere on or in the Ellipse.

9 The function $F(x, y) = x^2y - 4xy + 3x^2 + \frac{1}{2}y^2$ has three critical points, at $x = 0$, $x = 1$, and $x = 5$.

(a) Find the values of y at these three critical points.

Solution: Recall that critical points are points where $\nabla F = \mathbf{0}$. In this case, $\nabla F = \langle 2xy - 4y + 6x, x^2 - 4x + y\rangle$. We can simply find ∇F at the three x-values given:

$$\nabla F(0, y) = \langle -4y, y\rangle = \langle 0, 0\rangle \qquad \text{when } y = 0$$
$$\nabla F(1, y) = \langle 6 - 2y, y - 3\rangle = \langle 0, 0\rangle \qquad \text{when } y = 3$$
$$\nabla F(5, y) = \langle 6y + 30, y + 5\rangle = \langle 0, 0\rangle \qquad \text{when } y = -5.$$

Thus our critical points are $(0, 0)$, $(1, 3)$, and $(5, -5)$.

(b) Classify each critical point as a maximum, minimum, or saddle point.

Solution: Here $D = F_{xx}F_{yy} - F_{xy}^2 = (2y + 6)(1) - (2x - 4)^2$, so we get the following classification:

Critical Point	$(0, 0)$	$(1, 3)$	$(5, -5)$
Value of $F(x, y)$	0	$-\frac{3}{2}$	$\frac{125}{2}$
D	-10	8	-40
f_{xx}	6	12	-4
Classification	Saddle	Local Min	Saddle

10 Rectangular (Cartesian) and polar coordinates are related by the equations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

Suppose that at a certain instant in time, a particle is located at $(x, y) = (3, 4)$, and its polar coordinates are changing as specified by $\frac{dr}{dt} = 2$ and $\frac{d\theta}{dt} = 1$. Use the chain rule to calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for the particle at this instant.

Solution: The chain rule in this case says

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} \qquad \text{and} \qquad \frac{dy}{dt} = \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt}.$$

We compute these derivatives (using the fact that at $(x, y) = (3, 4)$, we have $r = 5$ and $\cos(\theta) = \frac{3}{5}$, $\sin(\theta) = \frac{4}{5}$) to get:

$$\frac{dx}{dt} = \cos(\theta)\frac{dr}{dt} - r\sin(\theta)\frac{d\theta}{dt} = \frac{3}{5}\cdot 2 - 4\cdot 1 = -\frac{14}{5}$$

and

$$\frac{dy}{dt} = \sin(\theta)\frac{dr}{dt} + r\cos(\theta)\frac{d\theta}{dt} = \frac{4}{5}\cdot 2 + 3\cdot 1 = \frac{23}{5}.$$

11 (a) Using Cartesian coordinates, evaluate the integral of the function x^2+y^2 over the right triangle with vertices $(x,y) = (0,0)$, $(a,0)$, and (a,a).

Solution: This is

$$\int_0^a \int_0^x \left(x^2+y^2\right)\, dy\, dx = \int_0^a \left[x^2 y + \frac{1}{3}y^3\right]_0^x dx = \frac{4}{3}\int_0^a x^3\, dx$$
$$= \frac{4}{3}\cdot\frac{1}{4}x^4\Big|_0^a = \frac{a^4}{3}.$$

(b) Evaluate the same integral using polar coordinates.
Hint: Make the substitution $u = \tan\theta$.

Solution: The first trick here is to write the line $x = a$ in polar coordinates as $r = a\sec(\theta)$. Then the integral is

$$\int_0^{\pi/4}\int_0^{a\sec(\theta)} r^2\cdot r\, dr\, d\theta = \int_0^{\pi/4} \frac{1}{4}a^4\sec^4(\theta)\, d\theta = \frac{a^4}{4}\int_0^{\pi/4}\sec^4(\theta)\, d\theta.$$

Now we make the substitution $u = \tan(\theta)$, so $du = \sec^2(\theta)$ and $\sec^2(\theta) = 1+\tan^2(\theta) = 1+u^2$:

$$\int_0^{\pi/4}\int_0^{a\sec(\theta)} r^2\cdot = \frac{a^4}{4}\int_0^{\pi/4}\sec^4(\theta)\, d\theta = \frac{a^4}{4}\int_0^{\pi/4}\sec^2(\theta)\ \sec^2(\theta)\, d\theta$$
$$= \frac{a^4}{4}\int_0^1 (1+u^2)\, du$$
$$= \frac{a^4}{4}\left[u+\frac{1}{3}u^3\right]_0^1 = \frac{a^4}{4}\cdot\frac{4}{3} = \frac{a^4}{3},$$

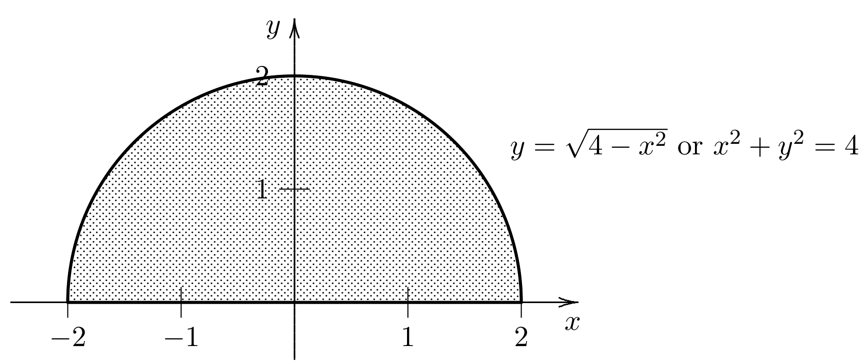
as before.

12 Convert the integral

$$\int_{-2}^2\int_0^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\, dy\, dx$$

to polar coordinates, then evaluate it exactly. Sketch the region R over which the integration is being performed.

Solution: Here's a quick sketch of the region of integration:



In polar coordinates this integral is

$$\int_{-2}^2\int_0^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\, dy\, dx = \int_0^{\pi}\int_0^2 e^{-r^2}\, r\, dr\, d\theta = \frac{\pi}{2}\left(1-e^{-4}\right).$$

13 Find the volume of the solid inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 1$, and below the plane $x + y + z = 5$.

Solution: This volume is the double integral

$$V = \iint_R [(5 - x - y) - 1] \; dA,$$

where R is the region inside the circle $x^2 + y^2 = 4$ in the plane. This seems a likely candidate for integration in polar coordinates, so we write this as

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 (4 - r\cos(\theta) - r\sin(\theta)) \; r \; dr \; d\theta = \int_0^{2\pi} \int_0^2 \left(4r - r^2\cos(\theta) - r^2\sin(\theta)\right) \; dr \; d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{3}r^3\cos(\theta) - \frac{1}{3}r^3\sin(\theta)\right]_0^2 \; d\theta = \int_0^{2\pi} \left(8 - \frac{8}{3}\cos(\theta) - \frac{8}{3}\sin(\theta)\right) \; d\theta \\
&= \left[8\theta - \frac{8}{3}\sin(\theta) + \frac{8}{3}\cos(\theta)\right]_0^{2\pi} = 16\pi.
\end{aligned}$$