Third Old Exam Two

1 Suppose $F(x,y) = 2e^{x-y} + 2$.

(a) Write down an equation for the tangent plane to the graph of $F(x,y)$ at the point where $x = y = 10$.

(b) Estimate the value of $F(10.1, 10.2)$ to one decimal place using the technique of linear approximation.

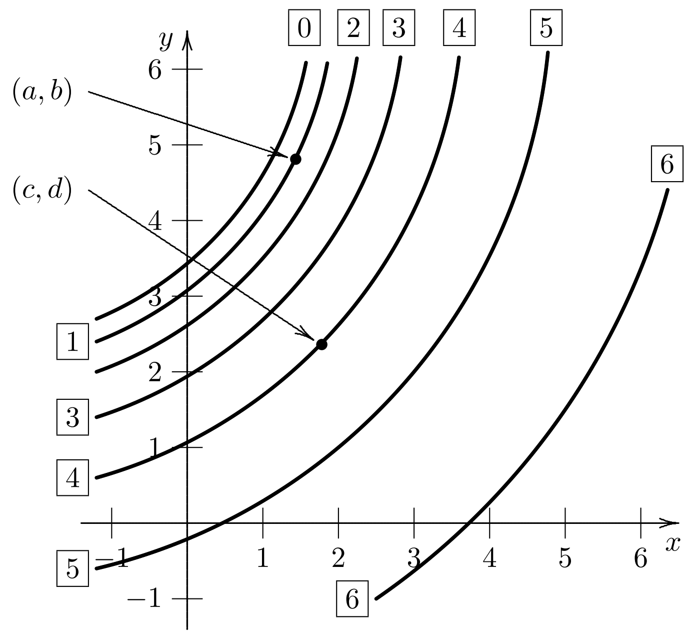
2 Consider the function $F(x,y) = x^2 + y^2 + x^2 y$.

(a) Find all the critical points of the function $F(x,y)$.

(b) For each of the critical points you found in part (a), decide whether it is a local maximum, local minimum, or a saddle for $F(x,y)$.

(c) Find the maximum value of $F(x,y)$ on the triangular region with vertices $(0,0)$, $(0,-2)$, and $(2,-2)$. The region includes the boundary of the triangle.

3 The contour plot of the function $f(x,y)$ is given in the figure below.



(a) Indicate whether the statements below are true or false, and give an explanation.

(i) $f_x(a,b) \neq 0$

(ii) $f_x(a,b) > f_x(c,d)$

(iii) $f_x(a,b) < f_y(a,b)$

(iv) $f_{xx}(a,b) > 0$

(b) At each of the labeled points, draw a unit vector in the direction of ∇f at the point.

(c) Which vector is longer, $\nabla f(a,b)$ or $\nabla f(c,d)$?

4 A moth's position at time t seconds is given by the position vector $\mathbf{r}(t) = \langle \cos(\pi t), t\sin(\pi t), 10 \rangle$ (for $t \geq 0$). Suppose the temperature at any point in space is given in degrees Fahrenheit by $T(x,y,z) = 2xz + y^2 + z^3 + 40$. What is the rate of change of the temperature (in degrees per second) as experienced by the moth at time $t = 1$?

5 Let $F(x,y) = g(x^2 y)$ where g is a continuous function of one variable with continuous first and second derivatives. Calculate $F_{xy}(2,2) + F_{yx}(2,2)$ if you also know that $g'(8) = 3$ and $g''(8) = 1$.

6 Suppose $F(x,y)$ is a function such that $D_{\mathbf{u}}F(1,1) = 5$ and $D_{\mathbf{v}}F(1,1) = \sqrt{2}$, where $\mathbf{u} = \langle 1, 0\rangle$ and $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(a) Find the vector $\nabla F(1,1)$.

(b) What is the maximum possible value of $D_{\mathbf{w}}F(1,1)$ for a unit vector $\mathbf{w}$?

(c) Find a unit vector $\mathbf{w}$ such that $D_{\mathbf{w}}F(1,1) = 0$.

7 Captain Kurt of the starship Interprize has steered his starship toward a wormhole in space. The Interprize's current location is $(0, 2, 2)$. The wormhole is quite oddly shaped, and its surface is given by the equation $-x^2 + 2y^2 + 2z^2 = 1$. Captain Kurt is really eager to get to the wormhole as soon as possible. Find a point on the wormhole's surface that is nearest to the Interprize so that Kurt knows where to send the ship. Be sure to write up your solution to this problem carefully, explaining all the steps involved in getting to your answer.

8 A battle to save a wildlife preserve in Alaska has led to Congress authorizing drilling for oil on the Ellipse, a park between the White House and the Washington Monument bounded by the curve $x^2 + 4y^2 = 100$. Geologists have studied the matter and report that the value V of the oil from a well drilled on the Ellipse will be given by the formula $V = 200 + 18y - x^2 - y^2$.

(a) The President asks you, the Secretary of Energy, to find the coordinates (x, y) of the most valuable drilling site, the least valuable drilling site, and the maximum and minimum values. Do so.

(b) The President has a follow-up question: is there a more valuable site if drilling was allowed outside the ellipse? Determine if there is a drilling site with even greater V and, if so, where it is located.

9 The function $F(x,y) = x^2y - 4xy + 3x^2 + \frac{1}{2}y^2$ has three critical points, at $x = 0$, $x = 1$, and $x = 5$.

(a) Find the values of y at these three critical points.

(b) Classify each critical point as a maximum, minimum, or saddle point.

10 Rectangular (Cartesian) and polar coordinates are related by the equations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

Suppose that at a certain instant in time, a particle is located at $(x, y) = (3, 4)$, and its polar coordinates are changing as specified by $\frac{dr}{dt} = 2$ and $\frac{d\theta}{dt} = 1$. Use the chain rule to calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for the particle at this instant.

11 (a) Using Cartesian coordinates, evaluate the integral of the function $x^2 + y^2$ over the right triangle with vertices $(x, y) = (0, 0)$, $(a, 0)$, and (a, a).

(b) Evaluate the same integral using polar coordinates.
Hint: Make the substitution $u = \tan\theta$.

12 Convert the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dy\,dx$$

to polar coordinates, then evaluate it exactly. Sketch the region R over which the integration is being performed.

13 Find the volume of the solid inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 1$, and below the plane $x + y + z = 5$.