1 (10 points) Find all critical points of $f(x,y) = x^2y - x^2 - 2y^2$, and classify each as a local minimum, local maximum, or saddle point.

Solution: Recall that the critical points are where $\nabla f = \mathbf{0}$, or $\langle f_x, f_y \rangle = \langle 0, 0 \rangle$. In this case, $\nabla f = \langle 2xy - 2x, x^2 - 4y \rangle$, so we get the equations $2x(y-1) = 0$ and $x^2 = 4y$. From the first equation, we have either $x = 0$ or $y = 1$. Plugging these into the second equation gives the points $(0,0)$ and $(\pm 2, 1)$.

We classify these points using the second derivative test. Since

$$f_{xx} = 2y - 2 \qquad f_{xy} = f_{yx} = 2x \qquad \text{and} \qquad f_{yy} = -4.$$

Thus $D = (2y-2)(-4) - (2x)^2 = 8 - 8y - 4x^2$ and we have the following classification:

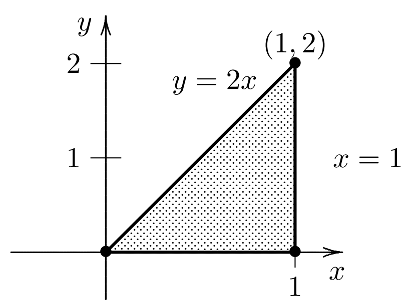
Critical Point	$f(x,y)$	D	f_{xx}	Classification
$(0,0)$	0	8	-2	Local Maximum
$(2,1)$	-2	-16	0	Saddle Point
$(-2,1)$	-2	-16	0	Saddle Point

2 (10 points) Consider the double integral

$$\int_0^2 \int_{y/2}^1 y e^{x^3} \, dx \, dy$$

(a) (3 points) Sketch the region of integration. Label all curves and all points of intersection and shade in the region.

Solution: The region is those values of x with $\frac{y}{2} \le x \le 1$ with $0 \le y \le 2$, as shown here:



(b) (4 points) Rewrite the integral using the order $dy\,dx$.

Solution: We can also think of this region as those points with $0 \le y \le 2x$ and x between 0 and 1:

$$\int_0^2 \int_{y/2}^1 y e^{x^3} \, dx \, dy = \int_0^1 \int_0^{2x} y e^{x^3} \, dy \, dx.$$

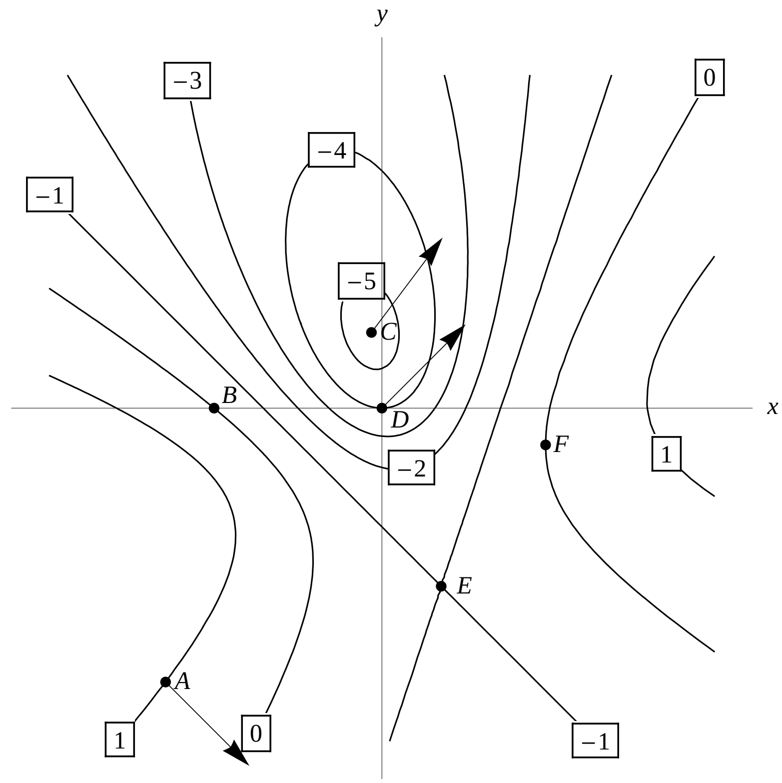
(c) (3 points) Compute the integral.

Solution: We can compute directly:

$$\int_0^1 \int_0^{2x} y e^{x^3} \, dy \, dx = \int_0^1 \left(\frac{y^2}{2} e^{x^3} \Big|_0^{2x} \right) dx = \int_0^1 2x^2 e^{x^3} \, dx$$

$$= \frac{2}{3} e^{x^3} \Big|_0^1 = \frac{2}{3}(e - 1).$$

3 (11 points) Here is the level set diagram (contour map) of a function $f(x,y)$. The value of f on each level set is indicated. Two of the six labeled points are critical points of f.



(a) (4 points) Which two points are critical points of f?

A $\quad B \quad$ **(C)** $\quad D \quad$ **(E)** $\quad F$

Classify each critical point as a local minimum, local maximum, or saddle point.

- Point C is a local minimum.
- Point E is a saddle point.

(b) (4 points) Two of the following four points have the property that $\frac{\partial f}{\partial x}$ is 0 at the point. Which two?

B $\quad$ **(D)** $\quad$ **(E)** $\quad F$

Solution: We remark that since the point E is a critical point – it was a saddle point in part (a) – it has $\nabla f = \mathbf{0}$, so $f_x = 0$ at E. One way to see that the derivative f_x is zero at D is by looking at the gradient. Since ∇f is vertical at D, the x-component f_x must be zero.

(c) (3 points) Decide whether the following directional derivatives are greater than 0, less than 0, or equal to 0. Circle the appropriate phrase.

Solution: Notice that we've drawn in the vectors $\mathbf{u}$ at the points A, C, and D.

i. $D_{\mathbf{u}}f(A)$, where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

greater than 0 $\qquad$ **(less than 0)** $\qquad$ equal to 0

Notice that $\mathbf{u}$ moves f from the $f=1$ level curve toward the $f=0$ level curve. Thus f is decreasing.

ii. $D_{\mathbf{u}}f(C)$, where $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

greater than 0   less than 0   **[equal to 0]**

Recall that C is a critical point, so $\nabla f = \mathbf{0}$. In particular this means that $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 0$, no matter what $\mathbf{u}$ is.

iii. $D_{\mathbf{u}}f(D)$, where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

greater than 0   **[less than 0]**   equal to 0

Notice that $\mathbf{u}$ moves f from the $f = -4$ level curve toward the $f = -5$ level curve. Thus f is decreasing.

4 (10 points) Let E be the solid consisting of all points (x, y, z) inside both the sphere $x^2+y^2+z^2 = 4$ and the cylinder $x^2 + y^2 = 1$. Find the volume of E.

Solution: This solid E is a cylinder (centered along the z-axis, radius 1) with spherical caps on the top and bottom. These caps are described by the functions $z = \sqrt{4 - x^2 - y^2}$ and $z = -\sqrt{4 - x^2 - y^2}$. Thus our volume will be

$$V = \iint_R \left(\sqrt{4 - x^2 - y^2} - \left(-\sqrt{4 - x^2 - y^2} \right) \right) \, dA$$

or, thinking geometrically that the entire volume is double the volume over the xy-plane,

$$V = 2 \iint_R \sqrt{4 - x^2 - y^2} \, dA.$$

Here the region R is the disk $x^2 + y^2 \leq 1$; this is simply the region in the xy-plane enclosed by the cylinder. Thus it's natural to use polar coordinates, in which $\sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$ and $dA = r \, dr \, d\theta$, so

$$V = 2 \int_0^{2\pi} \int_0^1 \sqrt{4 - r^2} \, r \, dr \, d\theta.$$

Using the substitution $u = 4 - r^2$, so $du = -2r \, dr$, we get

$$V = 2 \left(-\frac{1}{2} \right) \int_0^{2\pi} \int_{u=4}^{u=3} \sqrt{u} \, du \, d\theta = -\int_0^{2\pi} \left(\frac{2}{3} u^{3/2} \Big|_4^3 \right) d\theta = \frac{2}{3} \left(8 - 3\sqrt{3} \right) \int_0^{2\pi} d\theta = \frac{4\pi}{3} \left(8 - 3\sqrt{3} \right).$$

5 (10 points) Consider the surface given by $z = f(x, y)$, where $f(x, y) = x^4 + y^4 - 4xy + 10$.

(a) (3 points) At the point where $x = -1$ and $y = 2$, in which direction(s) is the height of the surface above the xy-plane *decreasing* most rapidly? Give your answer(s) in the form of a unit vector $\mathbf{u} = \langle a, b \rangle$.

Solution: We're interested in finding the vector $\mathbf{u}$ where $D_u f(-1, 2)$ (the directional derivative in the direction $\mathbf{u}$ at the point $(-1, 2)$) is smallest. Since $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$, this will be smallest when $\mathbf{u}$ is in the direction of $-\nabla f$.

Here $\nabla f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle$, so at the point $(x, y) = (-1, 2)$ we have $\nabla f(-1, 2) = \langle -12, 36 \rangle$. Thus we want $\mathbf{u}$ to be a unit vector in the direction of $-\nabla f(-1, 2) = \langle 12, -36 \rangle$. It's easier to first divide out a factor of 12: $\mathbf{u}$ is in the direction $\langle 1, -3 \rangle$, so $u = \left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle$.

(b) (3 points) At the point where $x = -1$ and $y = 2$, in which direction(s) is the height of the surface not changing at all? Give your answer(s) in the form of a unit vector $\mathbf{u} = \langle a, b\rangle$.

Solution: These directions would be where $D_{\mathbf{u}}f = 0$. Since $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$, this will happen when $\mathbf{u}$ is perpendicular to $\nabla f(1, -2) = \langle 12, -36\rangle$. The usual trick to find perpendicular vectors is to simply swap the terms and change one term's sign; we get two such directions: $\langle 36, 12\rangle$ and $\langle -36, -12\rangle$. As unit vectors, these are $u = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$ and $u = \left\langle -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$.

What if you don't know this trick to find orthogonal vectors? Well, we're looking for a unit vector $\mathbf{u} = \langle a, b\rangle$ such that $\langle 12, -36\rangle \cdot \langle a, b\rangle = 0$. This means $12a - 36b = 0$, or $a = 3b$. Thus $\mathbf{u} = \langle 3b, b\rangle$, so for this to be a unit vector we need $|b| = \frac{1}{\sqrt{10}}$. Thus $u = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$ ($b = +\frac{1}{\sqrt{10}}$) and $u = \left\langle -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$ ($b = -\frac{1}{\sqrt{10}}$), as before.

(c) (4 points) Use linear approximation to estimate $f(-0.8, 2.1)$.

Solution: Recall that the linear approximation to $f(x, y)$ near the point (x_0, y_0) is

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)\,.$$

(This is the approximation of $f(x, y)$ by its tangent plane at the point $(x_0, y_0, f(x_0, y_0))$. More on this later.) In our case, the point $(-0.8, 2.1)$ is close to the point $(x_0, y_0) = (-1, 2)$, so we use that. Since $f(-1, 2) = 35$, $f_x(-1, 2) = -12$, and $f_y(-1, 2) = 36$, we get

$$\begin{aligned} f(x, y) &\approx f(-1, 2) + f_x(-1, 2)\,(x + 1) + f_y(-1, 2)\,(y - 2) \\ &\approx 35 - 12\,(x + 1) + 36\,(y - 2) \end{aligned}$$

and so

$$f(-0.8, 2.1) \approx 35 - 12\,(-0.8 + 1) + 36\,(2.1 - 2) = 35 - 2.4 + 3.6 = 37.2.$$

The actual value is 36.5777.

In what sense is this approximation the tangent plane? The normal to the tangent plane to $g(x, y, z) = f(x, y) - z$ at the point $(x_0, y_0, f(x_0, y_0))$ is $\nabla g(x_0, y_0, z_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$; the tangent plane itself is

$$\langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle \cdot \langle x - x_0, y - y_0, z - f(x_0, y_0)\rangle = 0.$$

If we multiply this out and solve for z, this is

$$z = f(x_0, y_0) + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)\,.$$

Thus the approximation used above is simply that the height of the graph of the curve $z = f(x, y)$ is approximately the height of the graph of the tangent plane.

6 (12 points) Let $f(x, y) = x^2 - y^2$.

(a) (5 points) Let C be the level curve of f through the point $(4, 3)$. Find the tangent line to C at the point $(4, 3)$.

Solution: If we think of y as a function of x near the point $(4,3)$, then the slope of the requested tangent line is $\frac{dy}{dx}$. We find this by differentiating the level curve $f(x,y) = x^2 - y^2 = 7$ with respect to x; we get

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \qquad \text{or} \qquad 2x \cdot 1 - 2y \cdot \frac{dy}{dx} = 0.$$

Thus $\frac{dy}{dx} = \frac{x}{y}$, which is $\frac{4}{3}$ at the point $(x,y) = (4,3)$.

The tangent line thus has slope $\frac{4}{3}$ and passes through the point $(4,3)$; it is

$$y - 3 = \frac{4}{3}(x-4) \qquad \text{or} \qquad y = \frac{4}{3}x - \frac{7}{3} \qquad \text{or} \qquad 4x - 3y = 7.$$

Another way to do this is notice that the tangent line is perpendicular to the gradient vector $\nabla f = \langle 2x, -2y \rangle = \langle 8, -6 \rangle$. Thus the point (x,y) lies on the tangent line through $(4,3)$ exactly when the vector $\langle x-4, y-3 \rangle$ is perpendicular to this gradient. So we get the equation

$$\langle 8, -6 \rangle \cdot \langle x-4, y-3 \rangle = 0 \qquad \text{or} \qquad 8(x-4) - 6(y-3) = 0 \qquad \text{or} \qquad 4x - 3y = 7,$$

as before.

(b) (3 points) Let $\mathbf{u}$ be the unit vector in the direction of $\langle 1, 2 \rangle$. Find the directional derivative $D_{\mathbf{u}}f$ at the point (x,y).

Solution: Recall that $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$ (we've normalized the vector to have length one) and $\nabla f = \langle 2x, -2y \rangle$. Thus $D_{\mathbf{u}}f = \langle 2x, -2y \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{2x-4y}{\sqrt{5}}$.

(c) (4 points) Let $\mathbf{u}$ be the unit vector in the direction of $\langle 1, 2 \rangle$. In the region $x^2 + y^2 \leq 4$, what is the maximum value of $D_{\mathbf{u}}f$?

Solution: We'd like to maximize $F(x,y) = D_{\mathbf{u}}f = \frac{2x-4y}{\sqrt{5}}$ on the closed bounded region $x^2 + y^2 \leq 4$. This is a two-part process: we need to check the interior critical points, then we also need to use Lagrange multipliers to check the points on the boundary.

Since $\nabla F = \left\langle \frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}} \right\rangle$, there are no critical points on the interior.

Now we wish to check for maximum points of $F(x,y)$ on the boundary $G(x,y) = x^2 + y^2 = 4$. These occur when $\nabla F = \lambda \nabla G$, or

$$\begin{aligned} \tfrac{2}{\sqrt{5}} &= \lambda 2x \\ -\tfrac{4}{\sqrt{5}} &= \lambda 2y. \end{aligned}$$

Thus $y = -2x$, which means that $x^2 + (-2x)^2 = 4$ or $x = \pm\frac{2}{\sqrt{5}}$. Thus we get two points: $(x,y) = \left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$ and $(x,y) = \left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$. Since

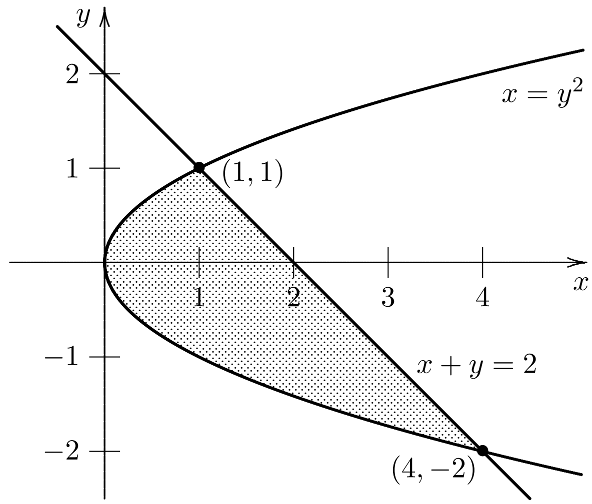
$$F\left(\tfrac{2}{\sqrt{5}}, -\tfrac{4}{\sqrt{5}}\right) = \frac{20/\sqrt{5}}{\sqrt{5}} = 4 \qquad \text{and} \qquad F\left(-\tfrac{2}{\sqrt{5}}, \tfrac{4}{\sqrt{5}}\right) = -\frac{20/\sqrt{5}}{\sqrt{5}} = -4,$$

we see that the maximum value of $F(x,y) = D_{\mathbf{u}}f$ is 4, which occurs at the point $(x,y) = \left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$.

7 (10 points) Let D be the region bounded by the parabola $x = y^2$ and the line $x + y = 2$.

(a) (4 points) Sketch the region D. Label all curves and all points of intersection and shade in the region D.

Solution: Here's a sketch of the two curves $x = y^2$ (which is $y = \pm\sqrt{x}$) and $x + y = 2$ (or $y = 2 - x$, if you prefer). The bounded region is shaded:



(b) (3 points) Write $\iint_D f(x,y)\,dA$ as an iterated integral (or a sum of iterated integrals) using the order $dx\,dy$.

Solution: Since we can write D as the set of points (x, y) with $y^2 \le x \le 2 - y$ (for y between -2 and 1), the double integral can be written as an iterated integral as follows:

$$\iint_D f(x,y)\,dA = \int_{-2}^{1}\int_{y^2}^{2-y} f(x,y)\,dx\,dy.$$

(c) (3 points) Write $\iint_D f(x,y)\,dA$ as an iterated integral (or a sum of iterated integrals) using the order $dy\,dx$.

Solution: We cannot write D as the set of points (x, y) with y bounded between two simple functions of x. For this reason we split it up into two regions: D_1 is the set of points (x, y) with $-\sqrt{x} \le y \le \sqrt{x}$ (and $0 \le x \le 1$) and D_2 is the set of points (x, y) with $-\sqrt{x} \le y \le 2 - x$ (and $1 \le x \le 4$). Thus

$$\begin{aligned}\iint_D f(x,y)\,dA &= \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA \\ &= \int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^4\int_{-\sqrt{x}}^{2-x} f(x,y)\,dy\,dx.\end{aligned}$$

8 (5 points) Indicate whether each statement is true or false. No explanations are required.

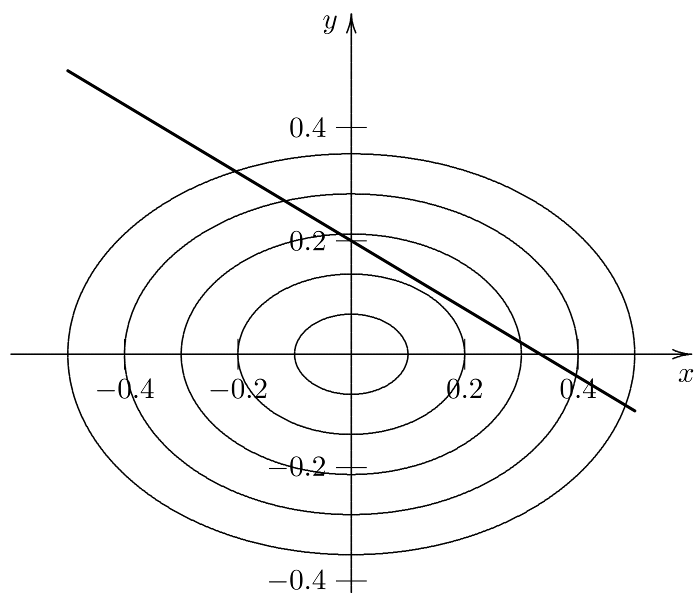
(a) **T F** If (a, b) is a critical point of $f(x, y)$ such that $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) and $f_{yy}(a, b) > 0$, then (a, b) must be a local minimum of $f(x, y)$. (You may assume that f and all of its derivatives are continuous.)

Solution: This is **True**. That $D > 0$ means that (a, b) is either a maximum or a minimum. The customary test is to check $f_{xx}(a, b)$, the concavity of $f(x, y)$ when restricted to the $y = b$ trace (parallel to the xz-plane). So if $f_{xx}(a, b) > 0$, for example, then $f(x, y)$ is concave up in the x direction and so (a, b) is a minimum rather than a maximum. But there's no reason we need to look at $f_{xx}(a, b)$ rather than $f_{yy}(a, b)$; certainly if $f(x, y)$ is concave up in the y direction, then (a, b) can only be a minimum (and not a maximum).

Another way to look at this is algebraic. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(x, y) = (a, b)$, then $f_{xx}f_{yy} > f_{xy}^2 \geq 0$. In particular, f_{xx} and f_{yy} must be the same sign, so $f_{yy}(a, b) > 0$ means $f_{xx}(a, b) > 0$ as well. Thus (a, b) is a minimum.

(b) **T F** The function $f(x, y) = x^2 + 2y^2$ attains an absolute minimum on $3x + 5y = 1$.

Solution: This is **True**. This is a typical Lagrange multipliers problem, and it can be helpful to visualize this geometrically:



The statement says there is a smallest ellipse (level curve of $f(x, y) = x^2 + 2y^2$) that touches the line $3x + 5y = 1$. From the picture we see that this is true.

(c) **T F** The function $f(x, y) = x^2 + 2y^2$ attains an absolute maximum on $3x + 5y = 1$.

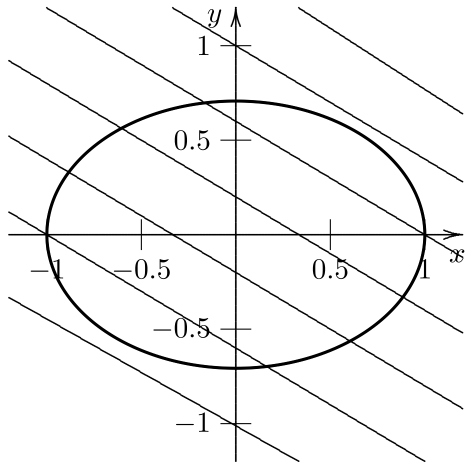
Solution: This is **False**. If we're restricted to the region $3x + 5y = 1$, then $y = \frac{1}{5} - \frac{3}{5}x$ goes to $-\infty$ as x grows without bound. In particular, $f(x, y)$ is unbounded on this set.

We could also use the visualization from the previous part. In this case our statement says that there is a *largest* ellipse that touches this line. This is clearly false from the picture.

(d) **T F** The function $f(x, y) = 3x + 5y$ attains an absolute minimum on $x^2 + 2y^2 = 1$.

Solution: This is **True**. In this case the set of point with $x^2 + 2y^2 = 1$ is a closed and bounded set, so the continuous function $f(x, y) = 3x + 5y$ must attain both an absolute maximum and an absolute minimum on this set.

We can again visualize this geometrically:



The ellipse shown is $x^2 + 2y^2 = 1$, and the lines are level curves of $f(x, y) = 3x + 5y$ from $f(x, y) = -5$ to $f(x, y) = 7$. Clearly this function reaches an absolute maximum and an absolute minimum on the ellipse.

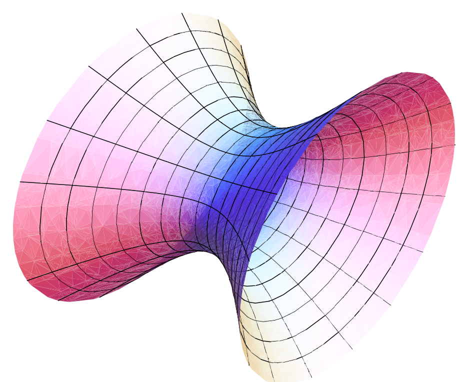
(e) **T F** The function $f(x, y) = 3x + 5y$ attains an absolute maximum on $x^2 + 2y^2 = 1$.

Solution: This is **True**. See the discussion from the previous statement.

9 (12 points) Consider the hyperboloid

$$2x^2 - y^2 + z^2 = 2$$

(a) (6 points) Find every point on the hyperboloid where the tangent plane is parallel to the plane $x - y - z = 0$.



Solution: The normal to the tangent plane of $f(x, y, z) = 2x^2 - y^2 + z^2 = 2$ is $\nabla f = \langle 4x, -2y, 2z \rangle$; the normal to the plane $x - y - z = 0$ is $\langle 1, -1, -1 \rangle$. These two planes are parallel when these normals are multiples of each other: $\langle 4x, -2y, 2z \rangle = C\langle 1, -1, -1 \rangle$. This means $C = 4x$, $C = 2y$, and $C = -2z$, so $y = 2x$ and $z = -2x$. Plugging these relationships into the equation for our hyperboloid, we get

$$2x^2 - (2x)^2 + (-2x)^2 = 2,$$

or $x^2 = 1$. Thus $x = \pm 1$, so there are two points: $(x, y, z) = (1, 2, -2))$ and $(x, y, z) = (-1, -2, 2))$.

(b) (6 points) Find the point (or points) on the hyperboloid closest to the origin.

Solution: The distance between a point (x, y, z) on the hyperboloid and the origin is $\sqrt{x^2 + y^2 + z^2}$. We wish to minimize this distance subject to the constraint that the point (x, y, z) lies on the hyperboloid $g(x, y, z) = 2x^2 - y^2 + z^2 = 2$ It simpler, however, to minimize $f(x, y, z) = x^2 + y^2 + z^2$. (Since the square root is an increasing function, this will give us the same points. Another way to see this is that the level surfaces of $\sqrt{x^2 + y^2 + z^2}$ are precisely the same as the level surfaces of $x^2 + y^2 + z^2$.)

We use the technique of Lagrange multipliers, so we wish to solve the equations $\nabla f = \lambda \nabla g$ and $g = 2$. These equations are

$$2x = 4x\lambda \quad (1)$$
$$2y = -2y\lambda \quad (2)$$
$$2z = 2z\lambda \quad (3)$$
$$2x^2 - y^2 + z^2 = 2.$$

The first three of these equations can be thought of as

$$\lambda = \frac{1}{2} \quad \text{or} \quad x = 0 \quad (1')$$
$$\lambda = -1 \quad \text{or} \quad y = 0 \quad (2')$$
$$\lambda = 1 \quad \text{or} \quad z = 0 \quad (3')$$

If $\lambda = \frac{1}{2}$, then equations $(2')$ and $(3')$ imply that $y = z = 0$. So the constraint means that $2x^2 - 0^2 + 0^2 = 2$ or $x = \pm 1$. Thus we get the points $(x, y, z) = (\pm 1, 0, 0)$; these points are at distance 1 from the origin.

If $\lambda = 1$, then equations $(1')$ and $(3')$ imply that $x = z = 0$. So the constraint means that $2(0)^2 - y^2 + 0^2 = 2$ or $y^2 = -2$. Thus there are no points on our hyperboloid with $x = z = 0$.

If $\lambda = -1$, then equations $(1')$ and $(2')$ imply that $x = y = 0$. So the constraint means that $2(0)^2 - 0^2 + z^2 = 2$ or $y = \pm\sqrt{2}$. Thus we get the points $(x, y, z) = (0, 0, \pm\sqrt{2})$; these points are at distance $\sqrt{2}$ from the origin.

Thus the points on the hyperboloid that are closest to the origin are $(\pm 1, 0, 0)$.

10 (10 points) Use a double integral to find the area bounded by one petal of the rosette

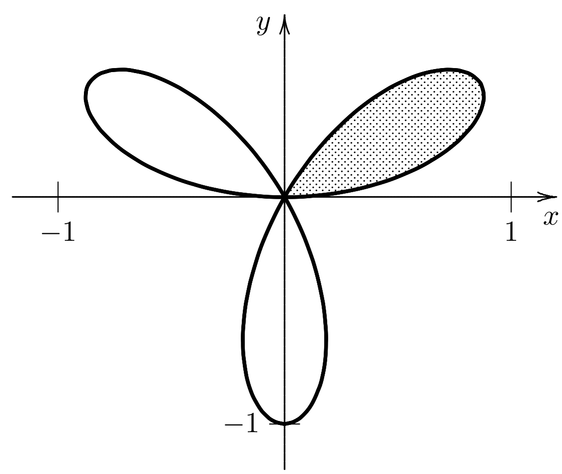
$$r = \sin(3\theta).$$

You may find it useful to recall that

$$\sin^2(x) = \frac{1}{2}\Big(1 - \cos(2x)\Big)$$

and

$$\cos^2(x) = \frac{1}{2}\Big(1 + \cos(2x)\Big).$$



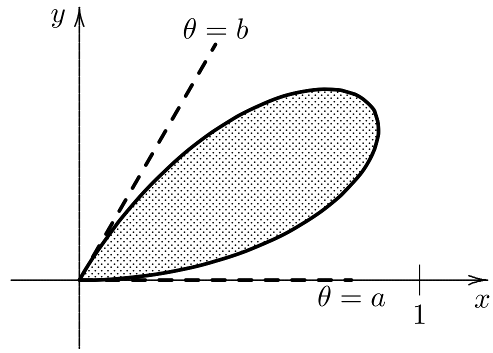
Solution: The area of a region D can be written as a double integral as

$$\iint_D 1 \, dA.$$

The region D is described for us in polar coordinates, so we should be aiming to describe this region as

$$D = \{(r, \theta) : 0 \le r \le \sin(3\theta),\ a \le \theta \le b\}.$$

But what are a and b (the limits on θ)? To help us figure this out, I'll sketch the picture again:



At the end points $\theta = a$ and $\theta = b$ we have $r = 0$. This means $\sin(3\theta) = 0$. Since $\sin(u) = 0$ when $u = 0$, π, 2π, or any multiple of π, we get $3\theta = 0$ or $3\theta = \pi$. This means the limits on θ are $\theta = 0$ to $\theta = \frac{\pi}{3}$. Thus the region is

$$D = \left\{(r, \theta) : 0 \le r \le \sin(3\theta),\ 0 \le \theta \le \tfrac{\pi}{3}\right\}.$$

This gives us our limits to turn our double integral into an iterated integral (being careful to use $dA = r\ dr\ d\theta$):

$$\text{Area} = \iint_D 1\ dA = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r\ dr\ d\theta$$
$$= \int_0^{\pi/3} \frac{1}{2} r^2 \Big|_0^{\sin(3\theta)} d\theta = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta)\ d\theta.$$

Now we use the double-angle formula from the hint to continue:

$$\text{Area} = \frac{1}{2} \int_0^{\pi/3} \frac{1}{2} \left(1 - \cos(6\theta)\right)\ d\theta = \frac{1}{4} \int_0^{\pi/3} \left(1 - \cos(6\theta)\right)\ d\theta$$
$$= \frac{1}{4} \left[\theta - \frac{1}{6} \sin(6\theta)\right]_0^{\pi/3}$$
$$= \frac{1}{4} \left[\left(\frac{\pi}{3} - \frac{1}{6} \sin(2\pi)\right) - \left(0 - \frac{1}{6} \sin(0)\right)\right]$$
$$= \frac{\pi}{12}.$$