Practice Problems for Weeks 5 & 6

1. (a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point $(1, 2, 1)$ on the surface $x^4 + y^4 + z^4 = 9x^2yz$.

    (b) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point $(1, 2, 1)$ on the surface $x + y + z = xy^2$.

2. A bee is buzzing around a room. At the moment that its location in cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{\pi}{3}, 1\right)$, its rate of change (with respect to time) in cylindrical coordinates is $\dfrac{dr}{dt} = 1$, $\dfrac{d\theta}{dt} = \pi$, and $\dfrac{dz}{dt} = -1$. Find the bee's rate of change in Cartesian coordinates at this moment; that is, find $\dfrac{dx}{dt}, \dfrac{dy}{dt}$, and $\dfrac{dz}{dt}$.

3. (a) Let $g(x)$ be a function of one variable such that $g(1) = 1$, $g(3) = -2$, $g'(1) = 5$, and $g'(3) = -7$. Let $f(x, y) = g(x^2y)$. Find $\frac{\partial f}{\partial x}(-1, 3)$ and $\frac{\partial f}{\partial y}(-1, 3)$.

    (b) Let $g(x, y)$ be a function of two variables with the following values:

	g	∇g
$(1, 2)$	1	$\langle 3, 5 \rangle$
$(2, 6)$	-1	$\langle -5, -2 \rangle$
$(0, 0)$	2	$\langle 4, 6 \rangle$

    Let $f(x, y, z) = g(xy, yz)$. Find $\frac{\partial f}{\partial x}(1, 2, 3)$, $\frac{\partial f}{\partial y}(1, 2, 3)$, and $\frac{\partial f}{\partial z}(1, 2, 3)$.

4. (a) Find the plane tangent to the graph of $f(x, y) = x^2 + \dfrac{y^2}{4}$ at the point $(1, 4, 5)$.

    (b) Use linear approximation to approximate $f(1.01, 4.01)$.

5. In each part, find the tangent plane to the surface at the indicated point.

    (a) $x^2 + y^2 - z^2 = 1$ at the point $(1, -1, 1)$.

    (b) The surface parameterized by $\vec{r}(u, v) = \langle u^3 - v^3, \sin(u + v), u^3 + v^3 \rangle$ at the point $(2, 0, 0)$.

    (c) The graph of $f(x, y) = x^4 + x^3y - xy^2$ at the point $(1, -1, -1)$.

    (d) The surface of the solid described by inequalities $0 \le x \le 6$ and $0 \le y^2 + z^2 \le 4$ at the point $(2, 1, \sqrt{3})$.

    (e) The surface of the solid described by inequalities $0 \le x \le 6$ and $0 \le y^2 + z^2 \le 4$ at the point $(6, 1, 1)$.

    (f) The surface described in cylindrical coordinates by $\theta = \frac{3\pi}{2}$ at the point $(r, \theta, z) = \left(2, \frac{3\pi}{2}, 5\right)$. (Give your answer in Cartesian coordinates.)

    (g) The surface described in spherical coordinates by $\phi = \frac{2\pi}{3}$ at the point $(\rho, \theta, \phi) = \left(2, \pi, \frac{2\pi}{3}\right)$. (Give your answer in Cartesian coordinates.)

6. Let $\mathcal{S}$ be the surface described parametrically by $\vec{r}(u, v) = \langle u \cos v, u \sin v, uv \rangle$, $0 \leq u, v \leq 4\pi$. Let $\mathcal{P}$ be the plane tangent to $\mathcal{S}$ at the point $\left(\pi, \sqrt{3}\pi, \frac{2\pi^2}{3} \right)$. Find a parameterization of $\mathcal{P}$.

7. The ellipsoid $x^2 + 4y^2 + 9z^2 = 36$ and the surface $z = \sin[\pi(x - y)]$ intersect in a curve, call it $\mathcal{C}$. Find the line tangent to $\mathcal{C}$ at the point $(6, 0, 0)$. (Please give a parametric vector equation for the line.)

8. You are skiing on a mountain which happens to be the graph of the function $f(x, y) = 10 - x^2 - y^4$. You are at the point $(1, 1, 8)$. If you want to ski down the steepest path, what direction should you head initially?

9. We have a function $f(x, y)$, and we are given the following information: $f_x(1, 2) = 1$ and $D_{\vec{u}}f(1, 2) = -5$, where $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

   (a) How many unit vectors $\vec{v}$ are there such that $D_{\vec{v}}f(1, 2) = 0$? Find all such vectors.

   (b) If $f(1, 2) = 3$, use linear approximation to approximate $f(1.06, 2.08)$.