Practice Problems for Week 7 (and Earlier)

1. True or false:

   (a) The level sets of the function $f(x, y, z) = x^2 + y^2$ are circles.

   (b) The graph of the function $f(x, y, z) = x^2 + y^2 + z^2$ is a sphere.

   (c) The level sets of the function $f(x, y, z) = x + y^2 + z^2$ are elliptic paraboloids.

2. *One of the most confusing topics in Math 21a is using gradients to find tangent planes of graphs of functions $f(x, y)$. This problem looks at a simpler, analogous case — using a gradient to find the tangent line of the graph of a function $f(x)$.*

   Consider the parabola $y = x^2$ (in $\mathbb{R}^2$). Use the fact that the gradient of a function is perpendicular to level sets of the function to find the line tangent to the parabola at $(3, 9)$. Be thorough and precise in your explanation: what function are you using the gradient of? How many variables is it a function of?

   Use single-variable calculus to check that you have the right equation for the tangent line.

3. We know two ways of finding the tangent plane for a surface. (Can you explain both of them?) Let S be a surface described as $z = f(x, y)$; that is, S is the graph of $f(x, y)$. Find the plane tangent to S at $(a, b, f(a, b))$ using both methods. Make sure your answers agree with each other.

4. In each part, $f(x, y)$ is a function of two variables and P is a point in $\mathbb{R}^2$. Based on the given information, decide which conclusion you can draw from the given information. Here are the choices:

   (A) P is a local minimum of f.

   (B) P is a local maximum of f.

   (C) P is a saddle point of f.

   (D) P is a critical point of f, but there is not enough information to determine what kind of critical point.

   (E) P is not a critical point of f.

   (F) There is not enough information to determine whether P is a critical point of f.

   Questions:

   (a) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = 1$, $f_{yy}(P) = 2$, and $f_{xy}(P) = 3$.

   (b) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = 1$, $f_{yy}(P) = 2$, and $f_{xy}(P) = 1$.

   (c) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = 2$, $f_{yy}(P) = -1$.

   (d) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = -2$, $f_{yy}(P) = -1$.

   (e) $f_x = 0$ and $D_{\vec{u}} f(P) = 1$, where $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.

   (f) $D_{\vec{u}} f(P) = 0$ and $D_{\vec{v}} f(P) = 0$, where $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ and $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(g) $D_{\vec{u}}f(P) = 0$ and $D_{\vec{v}}f(P) = 0$, where $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ and $\vec{v} = \langle -\frac{3}{5}, -\frac{4}{5} \rangle$.

(h) $D_{\vec{u}}f(P) = 0$, $D_{\vec{v}}f(P) = 0$, $[D_{\vec{u}}(D_{\vec{u}}f)](P) = 1$, and $[D_{\vec{v}}(D_{\vec{v}}f)](P) = 1$, where $\vec{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$ and $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

(i) $D_{\vec{u}}f(P) = 0$, $D_{\vec{v}}f(P) = 0$, $[D_{\vec{u}}(D_{\vec{u}}f)](P) = 1$, and $[D_{\vec{v}}(D_{\vec{v}}f)](P) = -1$, where $\vec{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$ and $\vec{v} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

(j) $f_{xx}(P) = 5$ and $[D_{\vec{u}}(D_{\vec{u}}f)](P) = -1$, where $\vec{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$.

5. (a) Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + 2z^2$ subject to the constraint that $x^2 + xy + y^2 + z^2 = 1$. (You may assume that the maximum and minimum values are in fact attained.)

   (b) Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + 2z^2$ subject to the constraint that $x^2 + y^2 + z^2 = 1$. (The maximum and minimum values are in fact attained: why?)

6. For which of the regions D described below is it true that every continuous function $f(x, y)$ must attain an absolute maximum value and absolute minimum value on D? (There may be more than one.)

   (a) D is the set of points (x, y) such that $|x| \leq 4$ and $|y| < 2$.

   (b) D is the set of points (x, y) such that $|x + y| \leq 1$.

   (c) D is the set of points (x, y) such that $x^2 + 4y^2 \leq 1$.

   (d) D is the set of points (x, y) such that $x^2 + 4y \leq 1$.

   (e) D is the set of points (x, y) such that $-x \leq y \leq x$.

   For each region D that you picked, find the absolute mininimum and absolute maximum value of $f(x, y) = x^2 - 4x + y^2$ on the region.

7. The temperature in a room is described by the function $T(x, y, z) = x^2y + z$. A bug is walking on a surface in the room, which can be described parametrically by $\vec{r}(u, v) = \langle u, e^v, u + v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq 1$. What is the warmest point the bug can reach? What is the coolest?