8/06/2002 GREENS THEOREM (Section 13.4-13.5) S-Math 21a

This is part 1 (of 3) of the weekly homework. It is due Aug 13 in the mailbox of Jon. More problems to this lecture can be found on pages 951-952 and 958-959 in the book.

SUMMARY.

- curl$(P,Q) = Q_x - P_y$. **Curl** for 2D vector fields.
- $\int_C (P,Q) \cdot dr = \int\int_D \text{curl}(P,Q)\, dA$ **Greens theorem**.
- $\int_C x\, dy$ **area** of D with boundary C.

1) (4 points) Evaluate the line integral of the vector field $F(x,y) = (x, y^2, x^2)$ along the rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$ in two ways. Do this by calculating the line integral

2) (4 points) Recalculate the line integral in the previous problem using Green's theorem.

3) (4 points) Calculate $F(x,y) = (x, y^2, x^2)$ along the rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$.

4) (4 points) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), then $\int_C xdy - ydx = x_1 y_2 - x_2 y_1$.

5) (4 points) Use the previous problem to verify that if $(x_1, y_1), ..., (x_n, y_n)$ are the vertices of a polygon, then $A = \frac{1}{2}(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + ... + (x_n y_1 - x_1 y_n)]$ is the area of the polygon.

CHALLENGE PROBLEM:

Let D be a region bounded by a simple closed path C in the plane. Use Green's theorem to prove that the coordinates of the center of mass are $(\int_C x^2 dy/(2A), -\int_C y^2 dx)/(2A)$, where A is the area of D.

SUPER CHALLENGE PROBLEM:

Verify that the planimeter calculates the area: Let $F(x,y) = (P(x,y), Q(x,y))$ be the planimeter vector field. It is defined by attaching a unit vector orthogonal to the vector $(x-a, y-b)$ at (x,y). The wheel rotation is the line integral of F along the boundary of R. **Green's theorem** tells that this integral is the double integral of $curl(F)$ over the region R. The planimeter vector field is explicitly given by $F(x,y) = (P(x,y), Q(x,y)) = (-(y - b(x,y)), (x - a(x,y)))$. Furthermore, $curl(F) = Q_x - P_y$ is equal to $2 + (-a_x - b_y)$ which is 2 plus the curl of the vector field $G(x,y) = (b(x,y), -a(x,y))$. Show that curl$(G) = -1$.