MATHs 21a CHECKLIST MATERIAL, 8/13/2003 — Maths 21a, Oliver Knill

- Here is a checklist of material we covered in this course.
- If you have worked with the material during the homework, there are not many things to memorize. An example might be the parameterization of the sphere or the definition of the curl.
- You don't need to know the formula for the curvature by heart but you should know what the curvature means and what is the curvature of a circle of radius r.
- The final exam will cover only material we mentioned **in class**. The handouts contain usually more material then what we covered in class.

1. Geometry of Space

- [ ] coordinates and vectors in the plane and in space
- [ ] $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$, $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
- [ ] dot product $v.w = v_1 w_1 + v_2 w_2 + v_3 w_3 = |v||w| \cos(\alpha)$
- [ ] cross product, $v.(v \times w) = 0$, $w.(v \times w) = 0$, $|v \times w| = |v||w| \sin(\alpha)$
- [ ] triple scalar product $u \cdot (v \times w)$ volume of parallelepiped
- [ ] parallel vectors $v \times w = 0$, orthogonal vectors $v \cdot w = 0$
- [ ] scalar projection $\text{comp}_w(v) = v \cdot w/|w|$
- [ ] vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$
- [ ] completion of square: example $x^2 - 4x + y^2 = 1$ is equivalent to $(x - 2)^2 + y^2 = -3$
- [ ] distance $d(P, Q) = |\vec{PQ}| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$

2. Lines, Planes, Functions

- [ ] symmetric equation of line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{z-z_0}{c}$
- [ ] plane $ax + by + cz = d$
- [ ] parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
- [ ] parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
- [ ] switch from parametric to implicit descriptions for lines and planes
- [ ] domain and range of functions $f(x, y)$
- [ ] graph $G = \{(x, y, f(x, y))\}$
- [ ] intercepts: intersections of G with coordinate axes
- [ ] traces: intersections with coordinate planes
- [ ] generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- [ ] quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, hyperboloid paraboloid
- [ ] plane $ax + by + cz = d$ has normal $\vec{n} = (a, b, c)$
- [ ] line $\frac{(x-x_0)}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $\vec{v} = (a, b, c)$
- [ ] sets $g(x, y, z) = c$ describe surfaces, example graphs $g(x, y, z) = z - f(x, y)$
- [ ] linear equation $2x + 3y + 5z = 7)$ defines plane
- [ ] quadratic equation i.e. $x^2 - 2y^2 + 3z^2 = 4$ defines quadric surface
- [ ] distance point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$
- [ ] distance point-line: $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$
- [ ] distance line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$
- [ ] finding plane through three points P, Q, R: find first normal vector

3. Curves

plane and space curves $\vec{r}(t)$
velocity $\vec{r}'(t)$, Acceleration $\vec{r}\,''(t)$
unit tangent vector $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$
unit normal vector $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$
binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
curvature $\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)|$
arc length $\int_a^b |\vec{r}\,'(t)|\, dt$
$\vec{r}\,'(t)$ is tangent to the curve
$\vec{v} = \vec{r}\,'$ then $\vec{r} = \int_0^t \vec{v}\, dt + \vec{c}$
$\kappa(t) = \frac{|r\,'(t) \times r\,''(t)|}{|r\,'(t)|^3}$
$\frac{d}{dt}(\vec{v}(t) \cdot \vec{w}(t)) = \vec{v}'(t) \cdot \vec{w}(t) + \vec{v}(t) \cdot \vec{w}'(t)$
T, N, B are unit vectors which are perpendicular to each other

4. Surfaces

polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$
cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$
spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
$g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
$g(r, \theta, z) = 0$ cylindrical surface, i.e. $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
$g(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
$f(x, y) = c$ level curves of $f(x, y)$
$g(x, y, z) = c$ level surfaces of $g(x, y, z)$
circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = (r\cos t, r\sin t)$.
ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\vec{r}(t) = (a\cos t, b\sin t)$
sphere: $x^2 + y^2 + z^2 = r^2$, $\vec{r}(u, v) = (r\cos u\sin v, r\sin u\sin v, r\cos v)$
ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\vec{r}(u, v) = (a\cos u\sin v, b\sin u\sin v, c\cos v)$
line: $ax + by = d$, $\vec{r}(t) = (t, d/b - ta/b)$
plane: $ax + by + cz = d$, $\vec{r}(u, v) = \vec{r}_0 + u\vec{v} + v\vec{w}, (a, b, c) = \vec{v} \times \vec{w}$
surface of revolution: $r(\theta, z) = f(z)$, $\vec{r}(u, v) = (f(v)\cos(u), f(v)\sin(u), v)$
graph: $g(x, y, z) = z - f(x, y) = 0$, $\vec{r}(u, v) = (u, v, f(u, v))$

5. Partial Derivatives

$f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
partial differential equation PDE: $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
$f_t = f_{xx}$ heat equation
$f_{tt} - f_{xx} = 0$ wave equation
$f_x - f_t = 0$ transport equation
$f_{xx} + f_{yy} = 0$ Laplace equation
$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
tangent line: $L(x, y) = L(x_0, y_0)$, $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0)$, $d = ax_0 + by_0$
tangent plane: $L(x, y, z) = L(x_0, y_0, z_0)$
estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
$f(x, y)$ differentiable if f_x, f_y are continuous
$f_{xy} = f_{yx}$ Clairot's theorem
$\vec{r}_u(u, v)$, $\vec{r}_v$ tangent to surface $\vec{r}(u, v)$

6. Gradient

$\nabla f(x,y) = (f_x, f_y)$, $\nabla f(x,y,z) = (f_x, f_y, f_z)$, gradient
$D_v f = \nabla i f \cdot v$ directional derivative
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ chain rule
$\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ which contains (x_0, y_0, z_0).
$\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f$ by chain rule
$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}$ normal line to surface $f(x,y,z) = c$ at (x_0, y_0, z_0)
$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$ tangent plane at (x_0, y_0, z_0)
directional derivative is maximal in the $\vec{v} = \nabla f$ direction
$f(x,y)$ increases, if we walk on the xy-plane in the ∇f direction
partial derivatives are special directional derivatives
if $D_v f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$
implicit differentiation: $f(x, y(x)) = 0$, $f_x 1 + f_y y'(x) = 0$ gives $y'(x) = -f_x/f_y$

7. Extrema

$\nabla f(x,y) = (0,0)$, critical point or stationary point
$D = f_{xx} f_{yy} - f_{xy}^2$ discriminant or Hessian determinant
$f(x_0, y_0) \geq f(x,y)$ in a neighborhood of (x_0, y_0) local maximum
$f(x_0, y_0) \leq f(x,y)$ in a neighborhood of (x_0, y_0) local minimum
$\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, λ Lagrange multiplier
two constraints: $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$
Second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ local max, $\nabla f = (0,0), D > 0, f_{xx} > 0$ local min, $\nabla f = (0,0), D < 0$ saddle

8. Double Integrals

$\int\int_R f(x,y)\; dA$ double integral
$\int_a^b \int_c^d f(x,y)\; dydx$ integral over rectangle
$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\; dydx$ type I region
$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\; dxdy$ type II region
$\int\int_R f(r,\theta) r\; drd\theta$ polar coordinates
$\int\int_R |\vec{r}_u \times \vec{r}_v|\; dudv$ surface area
$\int_a^b \int_c^d f(x,y)\; dydx = \int_c^d \int_a^b f(x,y)\; dxdy$ Fubini
$\int\int_R 1\; dxdy$ area of region R
$\int\int_R f(x,y)\; dxdy$ volume of solid bounded by graph(f) xy-plane

9. Triple Integrals

$\int\int\int_R f(x,y,z)\; dV$ triple integral
$\int_a^b \int_c^d \int_u^v f(x,y,z)\; dydx$ integral over rectangular box
$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\; dzdydx$ type I region
$f(r,\theta,z)\; \boxed{r}\; dzdrd\theta$ cylindrical coordinates
$\int\int\int_R f(\rho,\theta,z)\; \boxed{\rho^2 \sin(\phi)}\; dzdrd\theta$ spherical coordinates
$\int_a^b \int_c^d \int_u^v f(x,y,z)\; dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\; dxdydz$ Fubini
$V = \int\int\int_R 1\; dV$ volume of solid R
$M = \int\int\int_R \rho(x,y,z)\; dV$ mass of solid R with density ρ
$(\int\int\int_R x\; dV/V, \int\int\int_R y\; dV/V, \int\int\int_R z\; dV/V)$ center of mass

10. Line Integrals

$F(x,y) = (P(x,y), Q(x,y))$ vector field in the plane
$F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ vector field in space
$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$ line integral
$F(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative

11. Fundamental theorem of line integrals

FTL: $F(x,y) = \nabla f(x,y)$, $\int_a^b F(r(t)) \cdot r'(t))\, dt = f(r(b)) - f(r(a))$
For smooth gradient fields in the plane: $\int_C F\, dr = 0$, for all closed curves C

12. Green's Theorem

$F(x,y) = (P,Q)$, $\mathrm{curl}(F) = Q_x - P_y = \nabla \times F$.
Green's theorem: C boundary of R, then $\int_C F \cdot dr = \int\int_R \mathrm{curl}(F)\, dxdy$

12. Flux integrals

$F(x,y,z)$ vector field, $S = r(R)$ parametrized surface
$r_u \times r_v$ normal vector, $\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$ unit normal vector
$r_u \times r_v dudv = d\vec{S} = \vec{n} dS$ normal surface element
$\int\int_S F\, d\vec{S} = \int\int_S F(r(u,v)) \cdot (r_u \times r_v)\, dudv$ flux integral

13. Stokes Theorem

$F(x,y,z) = (P,Q,R)$, $\mathrm{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \nabla \times F$
Stokes's theorem: C boundary of surface S, then $\int_C F \cdot dr = \int\int_S \mathrm{curl}(F) \cdot dS$

14. Div Grad Curl

$\nabla = (\partial_x, \partial_y, \partial_z)$, $\mathrm{grad}(F) = \nabla f$,$\mathrm{curl}(F) = \nabla \times F$, $\mathrm{div}(F) = \nabla \cdot F$
$\mathrm{div}(\mathrm{curl}(F)) = 0$
$\mathrm{curl}(\mathrm{grad}(F)) = \vec{0}$

16. Divergence Theorem

$\mathrm{div}(P,Q,R) = P_x + Q_y + R_z = \nabla \cdot F$
Divergence theorem: E bounded by S then $\int\int_S F \cdot dS = \int\int\int_E \mathrm{div}(F)\, dV$