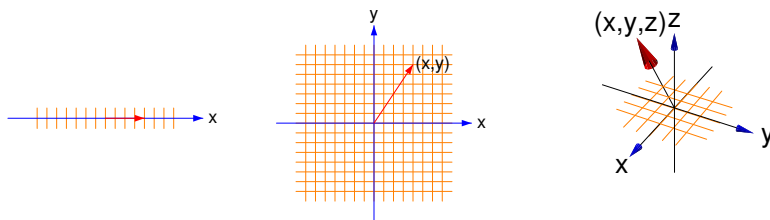


COORDINATES/DISTANCES

O. Knill, Maths21a

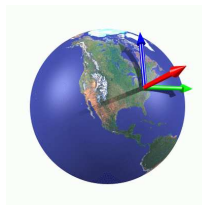
CARTESIAN COORDINATE SYSTEMS. A point on the line is labeled by a single coordinate x , a point in the **plane** is fixed by 2 coordinates (x, y) and a point in space is determined by three coordinates (x, y, z) . Depending on which coordinates are positive, one can divide the line, the plane or the space into **half lines**, **quadrants** or **octants**.

1D space = line = 2 half lines 2D space = plane = 4 quadrants 3D space = space = 8 octants



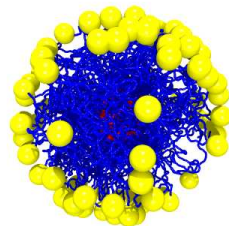
CHOICE OF COORDINATE SYSTEM.

Fixing the x, y or z -coordinate axis determines a **coordinate system** in space. The choice of the convenient coordinate system depends on the situation. On earth for example the coordinate system is chosen so that the z -axis points "up", perpendicular to the ground. But these directions depend on the position on earth of course.



EXAMPLE. Usually, we draw the coordinate system, such that the x, y coordinates are on the ground and the z -coordinate points "up". In two dimensions, on a sheet of paper, the x -coordinate usually is chosen to point "east" and the y -coordinate to point "north".

EXAMPLE. PHOTOGRAPHERS COORDINATE SYSTEM: In 3D graphics like computer games, virtual reality or ray tracing, it is custom to have the y -axis pointing up, the x -axis to the right and the z axis in front. This is the "**photographers coordinate system**". If the photo is the x - y plane, then the depth is the z axis. Is the photographers coordinate system a left or right-handed coordinate system?



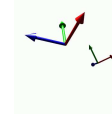
APPLICATION: Z-BUFFER. In computer graphics, the part of the memory reserved for storing the z -axis is called the "**z-buffer**". It is useful for "hidden line removal" in 2D rendering of a 3D scene: The z -axis is perpendicular to the screen with values increasing towards the viewer. Any point whose z -coordinate is smaller than the corresponding z -buffer value will be hidden behind parts which are already plotted.

DISTANCE. The **distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as

$$d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

(Pythagoras). While other distances could be defined like $d(P, Q) = |x-a| + |y-b| + |z-c|$, the Euclidean distance in the above formula is distinguished. Why?

PARITY. We usually work with a **right handed coordinate system**. The photographers coordinate system is an example of a left handed coordinate system. The "**right hand rule**": thumb= x -direction index finger= y -direction and middle finger= z -direction allows to check that the coordinate system is "right handed".



Parity is relevant in biology (orientation of DNA or Proteins) or particle physics, ("parity violation": physical laws change when we look at them in the mirror). Coordinate systems with different parity can not be rotated into each other.



GEOMETRICAL OBJECTS. **curves**, **surfaces** and **bodies** are examples of geometrical objects which can be described using **functions of several variables**. We look at some of them here to get some feel about space. The objects will be treated in more detail later.

SPHERE. A **sphere** of radius r centered at $P = (a, b, c)$ is the collection of points which have the distance r from P . The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

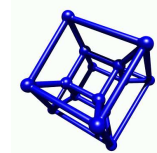
$$\begin{aligned} x^2 + bx + c &= 0 \\ x^2 + bx + (b/2)^2 &= (b/2)^2 - c \\ (x + b/2)^2 &= (b/2)^2 - c \\ x &= \pm\sqrt{(b/2)^2 - c} - b/2. \end{aligned}$$

COMPLETION OF SQUARE. The equation $x^2 + bx + c = 0$ is solved by adding $(b/2)^2 - c$ on both sides. This is the "**completion of the square**".

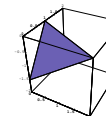
CENTER AND RADIUS OF A SPHERE. The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after completion of the square in each variable equivalent to $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. The equation describes therefore a sphere with **center** $(5/2, 1, 0)$ and **radius** $5/2$.



COORDINATE PLANES, QUADRANTS, OCTANTS. The coordinate axis $x = 0$, $y = 0$ divide the plane into 4 regions called **quadrants**. Similarly, the coordinate planes $x = 0$, $y = 0$ and $z = 0$ divide the space into 8 regions called **octants**. This could be continued into higher dimensions: how many "hyper-regions" are there in four dimensional "hyper-space" which is labeled by points with 4 coordinates (t, x, y, z) ? There are 16 hyper-regions and each of them contains one of the 16 points (x, y, z, w) , where x, y, z, w are either +1 or -1.



DESCRIBING PLANES. To draw the set of all points (x, y, z) which satisfy $x + 2y - 3z = 2$, we first find the intersections with the three coordinate axis. These **intersects** are $P = (2, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, -2/3)$. Then we draw the **traces**, the intersections of the set with the coordinate planes $x = 0$, $y = 0$ or $z = 0$. These three lines bound a triangle in space. Drawing this triangle indicates well the position of the plane.



HISTORICAL. In an appendix to "Geometry" to his "Discours de la methode" René Descartes (1596-1650) promoted the idea that algebra could be used as a general method to solve geometric problems. In honor of Descartes, the rectangular coordinate system is today called the **Cartesian coordinate system**.

Anectote: "In 1649, Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 a.m. and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace for 5 o'clock every morning, Descartes died of pneumonia.

