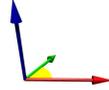


CROSS PRODUCT

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CROSS PRODUCT. The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ is defined as the vector $\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$. To compute it, look at the "determinant computation":



$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix} = i(v_2 w_3 - v_3 w_2) - j(v_1 w_3 - v_3 w_1) + k(v_1 w_2 - v_2 w_1).$$

DIRECTION OF $\vec{v} \times \vec{w}$: $\vec{v} \times \vec{w}$ is orthogonal to \vec{v} and orthogonal to \vec{w} .

Proof. Check that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

LENGTH: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$

Proof. The identity $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ can be proven by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|\cos(\alpha)$.

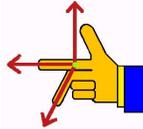
AREA. The length $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by \vec{v} and \vec{w} .

Proof. Because $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$, the area is $|\vec{v}||\vec{w}|\sin(\alpha)$ which is by the above formula equal to $|\vec{v} \times \vec{w}|$.

EXAMPLE. If $\vec{v} = (a, 0, 0)$ and $\vec{w} = (b \cos(\alpha), b \sin(\alpha), 0)$, then $\vec{v} \times \vec{w} = (0, 0, ab \sin(\alpha))$ which has length $|ab \sin(\alpha)|$.

ZERO CROSS PRODUCT. We see that $\vec{v} \times \vec{w}$ is zero if \vec{v} and \vec{w} are **parallel**.

ORIENTATION. The vectors \vec{v}, \vec{w} and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. The right hand rule is: put the first vector \vec{v} on the thumb, the second vector \vec{w} on the pointing finger and the third vector $\vec{v} \times \vec{w}$ on the third middle finger.



EXAMPLE. $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ forms a right handed coordinate system.

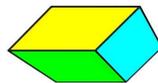
DOT PRODUCT (is a scalar)

CROSS PRODUCT (is a vector)

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$	commutative	$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$	anti-commutative
$ \vec{v} \cdot \vec{w} = \vec{v} \vec{w} \cos(\alpha)$	angle	$ \vec{v} \times \vec{w} = \vec{v} \vec{w} \sin(\alpha)$	angle
$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$	linearity	$(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$	linearity
$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$	distributivity	$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$	distributivity
$\{1, 2, 3\} \cdot \{3, 4, 5\}$	in Mathematica	Cross $\{1, 2, 3\}, \{3, 4, 5\}$	in Mathematica
$\frac{d}{dt}(\vec{v} \cdot \vec{w}) = \dot{\vec{v}} \cdot \vec{w} + \vec{v} \cdot \dot{\vec{w}}$	product rule	$\frac{d}{dt}(\vec{v} \times \vec{w}) = \dot{\vec{v}} \times \vec{w} + \vec{v} \times \dot{\vec{w}}$	product rule

TRIPLE SCALAR PRODUCT. The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.

PARALLELEPIPED. $[\vec{u}, \vec{v}, \vec{w}]$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ because $h = \vec{u} \cdot \vec{n}/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram which has area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = \vec{u} \cdot \vec{n}/|\vec{n}|/|\vec{n}| = \vec{u} \cdot (\vec{v} \times \vec{w})$.



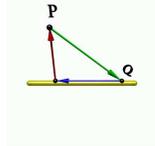
EXAMPLE. Find the volume of the parallel epiped which has the one corner $O = (1, 1, 0)$ and three corners $P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$ connected to it.

ANSWER: The parallelepiped is spanned by $\vec{u} = (1, 2, 1), \vec{v} = (3, 2, 1)$, and $\vec{w} = (0, 3, 2)$. We get $\vec{v} \times \vec{w} = (1, -6, 9)$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = -2$. The volume is 2.

DISTANCE POINT-LINE (3D). If P is a point in space and $L = Q + t\vec{u}$ be the line which contains the vector \vec{u} and the point Q , then

$$d(P, L) = |\vec{PQ} \times \vec{u}|/|\vec{u}|$$

is the distance between P and the line L . The representation of L as $r(t) = Q + t\vec{u}$ is called a **parametric equation** of the line.



PLANE THROUGH 3 POINTS P, Q, R :

The vector $\vec{n} = \vec{PQ} \times \vec{PR}$ is orthogonal to the plane. We will see next week that $\vec{n} = (a, b, c)$ defines the plane $ax + by + cz = d$, with $d = ax_0 + by_0 + cz_0$ which passes through the points $P = (x_0, y_0, z_0), Q, R$.

The cross product appears in many different applications:

ANGULAR MOMENTUM. If a mass point of mass m moves along a curve $\vec{r}(t)$, then the vector $\vec{L}(t) = m\vec{r}'(t) \times \vec{r}(t)$ is called the **angular momentum** of the point. It is coordinate system dependent.

ANGULAR MOMENTUM CONSERVATION.

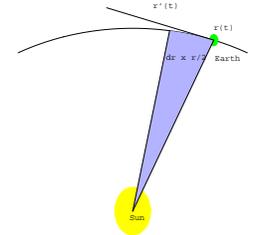
$$\frac{d}{dt}\vec{L}(t) = m\vec{r}'(t) \times \vec{r}''(t) + m\vec{r}(t) \times \vec{r}'''(t) = \vec{r}(t) \times \vec{F}(t)$$

In a central field, where $\vec{F}(t)$ is parallel to $\vec{r}(t)$, we get $d/dtL(t) = 0$ which means $L(t)$ is constant.

TORQUE. The quantity $\vec{r}(t) \times \vec{F}(t)$ is also called the **torque**. The time derivative of the **momentum** $p = m\vec{r}'$ is the **force** F . the time derivative of the **angular momentum** $\vec{L} = \vec{r}(t) \times \vec{p}(t) = m\vec{r}(t) \times \vec{r}'(t)$ is the **torque**.

KEPLER'S AREA LAW. (Proof by Newton)

The fact that $\vec{L}(t)$ is constant means first of all that $\vec{r}(t)$ stays in a plane spanned by $\vec{r}(0)$ and $\vec{r}'(0)$. The experimental fact that the vector $\vec{r}(t)$ sweeps over **equal areas in equal times** expresses angular momentum conservation: $|\vec{r}(t) \times \vec{r}'(t)dt/2| = |\vec{L}dt/m/2|$ is the area of a small triangle. The vector $\vec{r}(t)$ sweeps over an area $\int_0^T |\vec{L}dt/(2m) = |\vec{L}|T/(2m)$ in time $[0, T]$.



MORE PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS:

The **top**, the motion of a rigid body is describe by the angular momentum L and the angular velocity vector Ω in the body. Then $\dot{L} = L \times \Omega + M$, where M is an external **torque** obtained by external forces.

Electromagnetism: (informal) a particle moving along $\vec{r}(t)$ in a **magnetic field** \vec{B} for example experiences the force $\vec{F}(t) = q\vec{r}'(t) \times \vec{B}$, where q is the charge of the particle. In a constant magnetic field, the particles move on circles: if m is the mass of the particle, then $m\vec{r}''(t) = q\vec{r}'(t) \times \vec{B}$ implies $m\vec{r}'(t) = q\vec{r}(t) \times \vec{B}$. Now $d/dt|\vec{r}|^2 = 2\vec{r} \cdot \vec{r}' = \vec{r} \cdot q\vec{r}'(t) \times \vec{B} = 0$ so that $|\vec{r}|$ is constant.



Hurricanes are powerful storms with wind velocities of 74 miles per hour or more. On the northern hemisphere, hurricanes turn counterclockwise, on the southern hemisphere clockwise. This is a feature of all low pressure systems and can be explained by the Coriolis force. In a rotating coordinate system a particle of mass m moving along $\vec{r}(t)$ experience the following forces: $m\vec{\omega}' \times \vec{r}$ (inertia of rotation), $2m\vec{\omega} \times \vec{r}'$ (Coriolis force) and $m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ (Centrifugal force). The Coriolis force is also responsible for the circulation in Jupiter's Red Spot.

