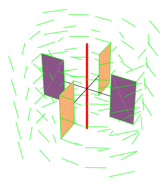
# CURL AND DIV — Maths21a, O. Knill

CURL (3D). The curl of a 3D vector field $F = (P, Q, R)$ is defined as the 3D vector field

$$\text{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) .$$

CURL (2D). Recall, the curl of a 2D vector field $F = (P, Q)$ is $Q_x - P_y$, a scalar field.



WANTED! Is there a multivariable calculus book, in which the above wheel is **not** shown? The wheel indicates the curl vector if $F$ is thought of as a wind velocity field. As we will see later the direction in which the wheel turns fastest, is the direction of curl($F$). The wheel could actually be used to measure the curl of the vector field at each point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl.

DIV (3D). The **divergence** of $F = (P, Q, R)$ is the scalar field $\text{div}(P, Q, R) = \nabla \cdot F = P_x + Q_y + R_z$.
DIV (2D). The **divergence** of a vector field $F = (P, Q)$ is $\text{div}(P, Q) = \nabla \cdot F = P_x + Q_y$.

NABLA CALCULUS. With the "vector" $\nabla = (\partial_x, \partial_y, \partial_z)$, we can write $\text{curl}(F) = \nabla \times F$ and $\text{div}(F) = \nabla \cdot F$. This is both true in 2D and 3D.

LAPLACE OPERATOR. $\Delta f = \text{divgrad}(f) = f_{xx} + f_{yy} + f_{zz}$ can be written as $\nabla^2 f$ because $\nabla \cdot (\nabla f) = \text{div}(\text{grad}(f)$. One can extend this to vectorfields $\Delta F = (\Delta P, \Delta Q, \Delta R)$ and writes $\nabla^2 F$.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with Nabla calculus from formulas for vectors like $\vec{v} \times \vec{v} = \vec{0}, \vec{v} \cdot \vec{v} \times \vec{w} = 0$ or $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$.

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|---|---|
| $\text{div}(\text{curl}(F)) = 0.$ | $\nabla \cdot \nabla \times F = 0.$ |
| $\text{curlgrad}(F) = \vec{0}$ | $\nabla \times \nabla F = \vec{0}.$ |
| $\text{curl}(\text{curl}(F)) = \text{grad}(\text{div}(F) - \Delta(F)).$ | $\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F.$ |

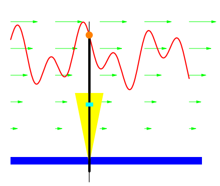
QUIZZ. Is there a vector field $G$ such that $F = (x + y, z, y^2) = \text{curl}(G)$? Answer: no because $\text{div}(F) = 1$ is incompatible with $\text{div}I\text{curl}(G)) = 0$.

ADDENDA TO GREEN'S THEOREM. Green's theorem, one of the advanced topics in this course is useful in physics. We have already seen the following applications

- Simplify computation of double integrals
- Simplify computation of line integrals
- Formulas for centroid of region
- Formulas for area
- Justifying, why mechanical integrators like the planimeter works.

APROPOS PLANIMETER.

The **cone planimeter** is a mechanical instrument to find the antiderivative of a function $f(x)$. It uses the fact that the vector field $F(x, y) = (y, 0)$ has $\text{curl}(F) = -1$. By Greens theorem, the line integral around the type I region bounded by 0 and the graph of $f(x)$ in the counter clockwise direction is $\int_a^x f(x)\ dx$. The planimeter determines that line integral.



THERMODYNAMICS. Gases or liquids are often described in a $P - V$ **diagram**, where the volume in the $x$-axis and the pressure in the $y$ axis. A periodic process like the pump in a refrigerator leads to closed curve $\gamma : r(t) = (V(t), p(t))$ in the $V - p$ plane. The curve is parameterized by the time $t$. At a given time, the gas has volume $V(t)$ and a pressure $p(t)$. Consider the vector field $F(V, p) = (p, 0)$ and a closed curve $\gamma$. What is $\int_\gamma F\ ds$? Writing it out, we get $\int_0^t (p(t), 0) \cdot (V'(t), p'(t)\ dt = \int_0^t p(t)V'(t)dt = \int_0^t pdV$. The curl of $F(V, p)$ is $-1$. You see by Green's theorem the integral $-\int_0^t pdV$ is the area of the region enclosed by the curve.

MAXWELL EQUATIONS (in homework you assume no current $j = 0$ and charges $\rho = 0$. $c$ = speed of light.

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|---|---|---|
| $\text{div}(B) = 0$ | No monopoles | there are no magnetic monopoles. |
| $\text{curl}(E) = -\frac{1}{c}B_t$ | Faraday's law | change of magnetic flux induces voltage |
| $\text{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$ | Ampère's law | current or change of E produces magnetic field |
| $\text{div}(E) = 4\pi\rho$ | Gauss law | electric charges are sources for electric field |

2D MAXWELL EQUATIONS? In space dimensions $d$ different than 3 the electromagnetic field has $d(d+1)/2$ components. In 2D, the magnetic field is a scalar field and the electric field $E = (P, Q)$ a vector field. The 2D Maxwell equations are $\text{curl}(E) = -\frac{1}{c}\frac{d}{dt}B, \text{div}(E) = 4\pi\rho$. Consider a region $R$ bounded by a wire $\gamma$. Green's theorem tells us that $d/dt \int\int_R B(t)\ dxdy$ is the line integral of $E$ around the boundary. But $\int_\gamma Eds$ is a voltage. A change of the magnetic field produces a voltage. This is the **dynamo** in 2D. We will see the real dynamo next week in 3D, where electromagnetism works (it would be difficult to generate a magnetic field in flatland). If $v$ is the velocity distribution of a fluid in the plane, then $\omega(x, y) = \text{curl}(v)(x, y)$ is the **vorticity** of the fluid. The integral $\int\int_R \omega dxdy$ is called the **vortex flux** through $R$. Green's theorem assures that this flux is related to the circulation on the boundary.

FLUID DYNAMICS. $v$ velocity, $\rho$ density of fluid.

| | | |
|---|---|---|
| Continuity equation | $\dot{\rho} + \text{div}(\rho v) = 0$ | no fluid get lost |
| Incompressibility | $\text{div}(v) = 0$ | incompressible fluids, in 2D: $v = \text{grad}(u)$ |
| Irrotational | $\text{curl}(v) = 0$ | irrotation fluids |

A RELATED THEOREM. If we rotate the vector field $F = (P, Q)$ by 90 degrees $= \pi/2$ we get a new vector field $G = (-Q, P)$. The integral $\int_\gamma F\ \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of $G$ through the boundary of $R$, where $dn$ is a normal vector with the length of $dr$. With $\text{div}(F) = (P_x + Q_y)$, we see that $\text{curl}(F) = \text{div}(G)$. Green's theorem now becomes

$$\int\int_R \text{div}(G)\ dxdy = \int_\gamma G \cdot dn ,$$

where $dn(x, y)$ is a normal vector at $(x, y)$ orthogonal to the velocity vector $r'(x, y)$ at $(x, y)$. This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. **It is in two dimensions just Green's theorem disguised.** There are only 2 basic integral theorems in the plane: Green's theorem and the FTLI.

PREVIEW. Green's theorem is of the form $\int_R F' = \int_{\delta R} F$, where $F'$ is a "derivative" and $\delta R$ is a "boundary". There are 2 such theorems in dimensions 2, three theorems in dimensions 3, four in dimension 4 etc. In the plane, Green's theorem is the second one besides the fundamental theorem of line integrals FTLI. In three dimensions, there are two more theorems beside the FTLI: Stokes and Gauss Theorems which we will see in the next week.

| dim | dim(R) | theorem |
|---|---|---|
| 1D | 1 | Fund. thm of calculus |

| | | |
|---|---|---|
| 2D | 1 | Fund. thm of line integrals |
| 2D | 2 | Green's theorem |

| dim | dim(R) | theorem |
|---|---|---|
| 3D | 1 | Fundam. thm of line integrals |
| 3D | 2 | Stokes theorem |
| 3D | 3 | Gauss theorem |

| | | |
|---|---|---|
| $1 \mapsto 1$ | $f'$ | derivative |
| $1 \mapsto 2$ | $\nabla f$ | gradient |
| $2 \mapsto 1$ | $\nabla \times F$ | curl |

| | | |
|---|---|---|
| $1 \mapsto 3$ | $\nabla f$ | gradient |
| $3 \mapsto 3$ | $\nabla \times F$ | curl |
| $3 \mapsto 1$ | $\nabla \cdot F$ | divergence |