CYLINDRICAL COORDINATES — Maths21a, O. Knill

REMINDER: INTEGRATION POLAR COORDINATES.

$$\int\int_R f(r,\theta)\ \boxed{r}\ d\theta dr \ .$$

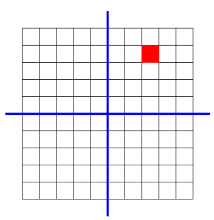
EXAMPLE 1. Area of a disk of radius R

$$\int_0^R \int_0^{2\pi} r\ d\theta dr = 2\pi \frac{r^2}{2}|_0^R = R^2\pi \ .$$

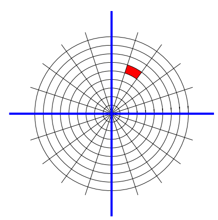
WHERE DOES THE FACTOR "r" COME FROM?

1. EXPLANATION. A small rectangle with dimensions $d\theta dr$ in the (r,θ) plane is mapped to a sector segment in the (x,y) plane. It has approximately the area $r d\theta dr$. It is small for small r.

2. EXPLANATION. The map $(r,\theta) \mapsto (r\cos(\theta, r\sin(\theta))(f(r,\theta), g(r,\theta))$ which changes from Cartesian to polar coordinates has the **Jacobean** is $T' = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix} = \begin{bmatrix} f_r & f_\theta \\ g_r & g_\theta \end{bmatrix}$ with determinant $f_r g_\theta - f_\theta g_r = r$. This is a special case of a more general formula which we do not cover in this class.



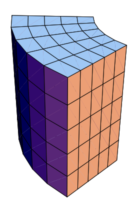
$T : (r,\theta,z) \mapsto (r\cos(\theta), r\sin(\theta), z)$



CYLINDRICAL COORDINATES. Use polar coordinates in the x-y plane and leave the z coordinate. Take $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$. The integration factor r is the same as in polar coordinates.

$$\int\int\int_{T(R)} f(x,y,z)\ dxdydz = \int\int\int_R g(r,\theta,z)\ \boxed{r}\ dr d\theta dz \ .$$

For example, if $f(x,y,z) = (x^2+y^2) + xz$, then $g(r,\theta,z) = r^2 + r\cos(\theta)z$.



EXAMPLE. Calculate the volume bounded by the parabolic $z = 1-(x^2+y^2)$ and the $x-y$ plane. In cylindrical coordinates, the paraboloid is given by the relation $z = 1 - r^2$:

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r\ dz d\theta dr = \int_0^1 \int_0^{2\pi} (r - r^3)\ d\theta dr = 2\pi(r^2/2 - r^4/4)_0^1 = \pi \ .$$

USE A GOOD PICTURE! A good conceptual picture not only helps to solve double and triple integral problems. Sometimes, it is even virtually impossible to solve the problem without having a good picture.

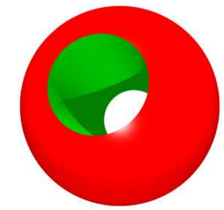
PROBLEM. Find the volume of the solid obtained by taking a sphere $x^2+y^2+z^2 = 1$ into which a hole $x^2+y^2 \leq 1/2$ has been drilled.

SOLUTION.

$$2\pi \int_{1/2}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr = 2\pi \int_{1/2}^1 2r\sqrt{1-r^2}\ dr$$

which is

$$-2\pi \frac{2}{3}(1-r^2)^{3/2}|_{1/2}^1 = \frac{4\pi}{3}\frac{\sqrt{27}}{8} = \pi\sqrt{3}/2$$

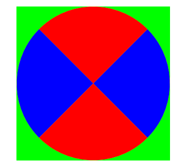


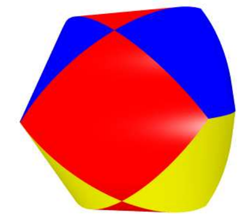
PROBLEM. Find the volume of the intersection of the three cylinders $x^2+y^2 \leq 1$, $x^2+z^2 \leq 1$ and $y^2+z^2 \leq 1$.

SOLUTION.

$$8\int_{-\pi/4}^{\pi/4} \int_0^1 \sqrt{1 - r^2\sin(\theta)^2}\ r dr d\theta$$

which is $-16/3 + 8\sqrt{2}$.





PROBLEM. Find the volume of the intersection of the two solid cylinders $x^2+y^2 \leq 1$ and $x^2+z^2 \leq 1$.

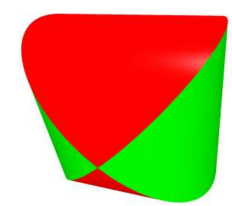
SOLUTION. We compute the volume in one of the 8 octants and multiply by 8 in the end.

$$8\int_0^1 \int_0^{\pi/2} \sqrt{1 - r^2\cos(\theta)^2}\ r d\theta dr = 16/3 \ .$$

Here is how one would evaluate this integral with Mathematica:

8 Integrate[Sqrt[1 − r^{2}Cos[theta]^2]r, {r, 0, 1}, {theta, 0, Pi/2}]

Note the order in which the integration range in entered the computer algebra system!



PROBLEM. Find $\int\int\int_R z^2\ dV$, where R of the solid obtained by intersecting $\{1 \leq x^2+y^2+z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2+y^2\}$.

SOLUTION. We split the integral up into a "cone part" $z \in [-\sqrt{2}, \sqrt{2}]$, and the cup part $|z| > \sqrt{2}$ and evaluate each separately. The double cone has the volume $2\pi \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^z r\ dr dz = 2\pi\sqrt{2}/3$. One cup has the volume $2\pi \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-z^2}} r\ dr dz = \pi \int_{\sqrt{2}}^2 (4-z^2)\ dz$. The total volume is $\pi(2-\sqrt{2})16/3$.