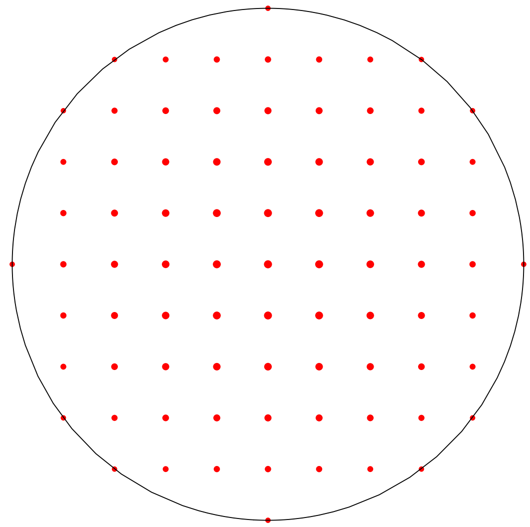
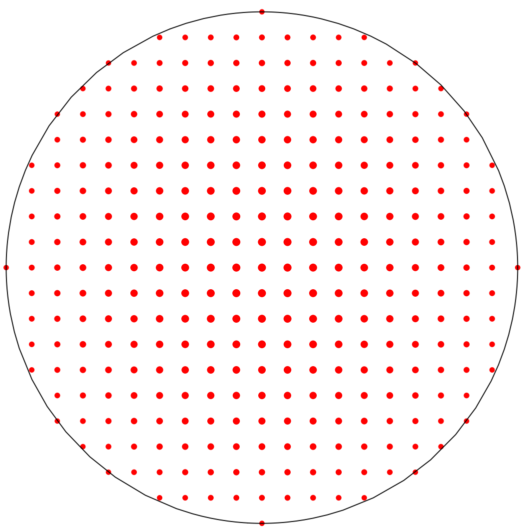
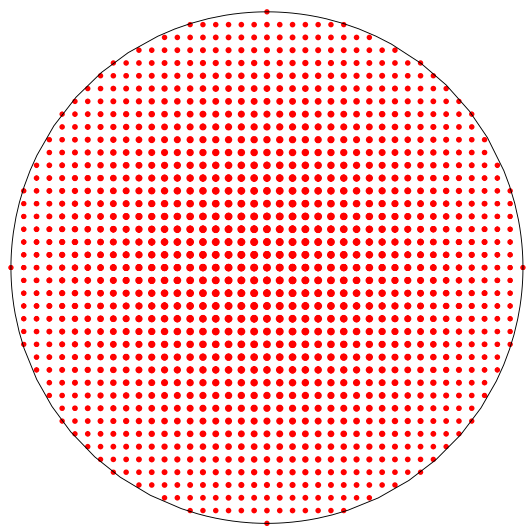
GAUSS CIRCLE PROBLEM **Maths21a, O. Knill**

We have defined the double integral of a function f over a region R as the limit $\frac{1}{n^2}\sum_{x_{ij}\in R} f(x_{ij})$, where $x_{ij} = (i/n, j/n)$. Let us look at the case of a **disc** of radius r and $n = 5$ and the case $f(x, y) = 1$, where we compute the area. What is the approximation area, which we get by counting the number of lattice points inside R?



The **Gauss circle problem** asks to estimate the number of lattice points $g(r) = \pi r^2 + E(r)$ enclosed by the circle r.
Here are some numbers $g(10) = 317, g(100) = 31417, g(1000) = 3141549, g(10000) = 314159053$.





Estimating the error $g(n)/n^2 - \pi$ is a famous open problem called the **"Gauss circle problem"**. If one writes $E(n) = g(n) - \pi n^2$, one believes that for every $\theta > 1/2$ there is a constant C such that $E(n) \leq C n^\theta$. Gauss knew already that this is true for $\theta = 1$. This has been improved over the 20'th century to $\theta = 46/73$.
Let's experiment with $\theta = 1/2$: $g(1000) - 1000^2\pi = -43.65$ is in absolute value less then $C = 2$ times $\sqrt{1000} = 31.6$, $g(10000) - 10000^2\pi = -212.359$ is in absolute value less then $C = 3$ times $\sqrt{10000} = 100$. Letting Mathematica count for a bit longer: $g(100'000) - 100'000^2\pi = -1078.9$ which is in absolute value smaller than $C = 3$ times $\sqrt{100000} = 316.2$.