GREEN'S THEOREM APPENDIX — Maths21a, O. Knill

GREEN'S THEOREM is extremely useful in physics.
It belongs to the most advanced topics in calculus. If you master it, you have so to speak the black belt in calculus. Of course, it needs some time to learn.

THERMODYNAMICS. When studying gases or liquids, one draws often a $P-V$ **diagram** (evenso one draws V usually as the x-axis). The volume in the x-axis and the pressure in the y axis.

For periodic (running of a car, the pump in a refrigerator), one gets a closed curve $\gamma : t \mapsto r(t) = (V(t), p(t))$ in the $V-p$ plane. The curve is parameterized by the time t. At a given time, the gas has a specific volume $V(t)$ and a specific pressure $p(t)$.

Consider the vector field $F(V,p) = (p,0)$ and a closed curve γ. What is $\int_\gamma F\ ds$? Writing it out, we get $\int_0^T (p(t),0) \cdot (V'(t), p'(t)\ dt = \int_0^T p(t)V'(t)dt = \int_0^T pdV$. The curl of $F(V,p)$ is -1. You see that the integral $-\int_0^T pdV$ is the area of the region enclosed by the curve.

Where is the physics? If the volume of a gas changes by constant pressure, then the work on the system is pdV. On the other hand, if the volume is held constant, then for a gas, one does no work on the system, when changing the pressure.

EXAMPLE: ELECTRO ENGINE. Let us look at a cyclic process, where the volume is decreased under low pressure and increased under high pressure. It is clear that the gas does some work in this case. How much is it? Well, it is $\int_0^T pdV$ for a curve which goes clockwise around a closed region R. Green's theorem tells us that the work done is the area of the region R.

ELECTROMAGNETISM. In the plane (flatland), the electric field is a vector field $E = (E_1, E_2)$, while the magnetic field is a scalar field. One of the 2D Maxwell equation is $\text{curl}(E) = -\frac{1}{c}\frac{d}{dt}B$, where c is the speed of light.

Consider a region R bounded by a wire γ. Greens theorem tells us that $d/dt \int\int_R B(t)dxdy$ is the line integral of E around the boundary. But $\int_\gamma Eds$ is a voltage. A change of the magnetic field produces a voltage. This is the **dynamo**.

This statement will become much more clear and useful when we lift Green to three dimensions (which will be called the theorem of Stokes).

FLUID DYNAMICS. If v is the velocity distribution of a fluid in the plane, then $\omega(x,y) = \text{curl}(v)(x,y)$ is called the **vorticity** of the fluid.

If $v = \nabla U$ (potential flows), then the fluid is called **irrotational**. As you can check, then $\omega = \text{curl}(v) = 0$ by Green's theorem.

The integral $\int\int_R \omega dxdy$ is called the vortex flux through R. Green's theorem tells that this flux is related to the circulation on the boundary.

In water, you could ideally measure the amount of vorticity inside a region R by measureing the work, you have to do by swimming around the boundary of R.

A RELATED THEOREM. If we rotate the vector field $F = (P,Q)$ by $\pi/2$ we get a new vector field $G = (-Q,P)$ and the integral $\int_\gamma F\ \cdot ds$ becomes a **flux** of G through the boundary of R. Introducing $\text{div}(F) = (P_x + Q_y)$ we see that $\text{curl}(F) = \text{div}(G)$. Greem's theorem now becomes

$$\int\int_R \text{div}(G)\ dxdy = \int_\gamma G \cdot dn\ ,$$

where $dn(x,y)$ is a normal vector at (x,y) orthogonal to the velocity vecgtor $r'(x,y)$ at a point (x,y).
This new theorem has a generalisation to three dimensions too. It is called Gauss's theorem or divergence theorem.