2D INTEGRAL APPLICATIONS — Maths21a, O. Knill

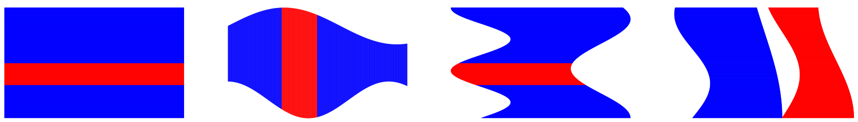
THINGS TO KEEP IN MIND.

- Double integrals can often be evaluated through iterated integrals

"Integrals have layers".

- $\iint_R 1 \; dxdy = \iint 1 \; dA$ is the **area** of the region R.
- $\iint_R f(x,y) \; dxdy$ is the volume of the solid having the graph of f as the "roof" and the R in the $xy-$ plane as the "floor".

TYPES OF REGIONS.
$\int\int_R f \; dA = \int_a^b \int_c^d f(x,y) \; dydx$ **rectangle**.
$\int\int_R f \; dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \; dydx$ **type I region**.
$\int\int_R f \; dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y) \; dxdy$ **type II region**.
A general region, we try to cut it into pieces, where each piece is a Type I or Type II region.



Rectangle Type I Type II To cut

AREA $A = \iint_R 1 \; dA$
MASS $M = \iint_R \rho(x,y) \; dA$
AVERAGE $\iint_R f(x,y) \; dA/A$.
CENTROID $(\iint_R x \; dA/A, \iint_R y \; dA/A)$.
CTR MASS $(\iint_R x\rho(x,y)dA, \iint_R y\rho(x,y)dA)/M$.
MOMENT OF INERTIA $I = \iint_R (x^2+y^2) \; dA$
RADIUS OF GYRATION $\sqrt{I/M}$

VOLUME $V = \iiint_R 1 \; dV$
MASS $M = \iint_R \rho(x,y,z) \; dV$
AVERAGE $\iiint_R f(x,y,z) \; dV/V$.
CENTROID $(\iiint_R x \; dV/V, \iiint_R y \; dV/V, \iiint_R z \; dV/V)$.
C.O.M. $(\iiint_R x\rho dV, \iiint_R x\rho dV, \iiint_R y\rho dV)/M$.
MOMENT OF INERTIA $I = \iiint_R (x^2+y^2) \; dV$
RADIUS OF GYRATION $\sqrt{I/M}$

AREA OF CIRCLE. To compute the area of the circle of radius r, we integrate

$$A = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \; dydx \; .$$

The inner integral is $2\sqrt{r^2-x^2}$ so that

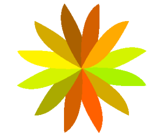
$$A = \int_{-r}^{r} 2\sqrt{r^2-x^2} \; dx \; .$$

This can be solved with a substitution: $x = r\sin(u), dx = r\cos(u)$. With the new bounds $a = -\pi/2, b = \pi/2$ and $\sqrt{r^2-x^2} = \sqrt{r^2 - r^2\sin^2(u)} = r\cos(u)$ we end up with

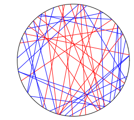
$$A = \int_{-\pi/2}^{\pi/2} 2r^2\cos^2(u) \; du = \int_{-\pi/2}^{\pi/2} r^2(1+\cos(2u)) \; du = r^2\pi \; .$$

MOMENT OF INERTIA. Compute the kinetic energy of a square iron plate $R = [-1,1]\times[-1,1]$ of density $\rho = 1$ (about 10cm thick) rotating around its center with a $6'000 rpm$ (rounds per minute). The angular velocity speed is $\omega = 2\pi \; 6'000/60 = 100 \; 2\pi$. Because $E = \int\int_R (r\omega)^2/2 \; dxdy$, where $r = \sqrt{x^2+y^2}$, we have $\boxed{E = \omega^2 I/2}$, where $I = \int\int_R (x^2+y^2) \; dxdy$ is the **moment of inertia**. For the square, $I = 8/3$. Its energy of the plate is $\omega^2 8/6 = 4\pi^2 100^2 8/6 Joule \sim 0.86 KWh$. You can run with this energy a 60 Watt bulb for 14 hours.

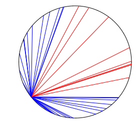
A FLOWER. What is $\int\int_R x^2+y^2 \; dxdy$, where R is a flower obtained by rotating the region enclosed by the curves $y = x^2$ and $y = 2x - x^2$ by adding multiples of the angles $2\pi/12$?
SOLUTION. The moment of inertia of all the petals add up: $I = 12\int_0^1 \int_{x^2}^{2x-x^2} (x^2+y^2) \; dydx = \int_0^1 [x^2y + y^3/3]_{y=x^2}^{y=2x-x^2} \; dx = 1243/210 = 86/35$.



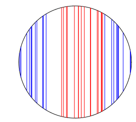
PROBLEM: BERTRAND'S PARADOX (Bertrand 1889)
We throw randomly lines onto the disc. What is the probability that the intersection with the disc is larger than the length $\sqrt{3}$ of the equilateral triangle inscribed in the unit circle?



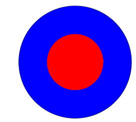
Answer Nr 1: take an arbitrary point P in the disc. The set of lines which pass through that point is parametrized by an angle ϕ. In order that the chord is longer than $\sqrt{3}$, the angle has to fall within an angle of 60° of a total of 180°. The probability is $\boxed{1/3}$.



Answer Nr 2: consider all lines perpendicular to a fixed diameter. The chord is longer than $\sqrt{3}$, when the point of intersection is located on the middle half of the diameter. The probability is $\boxed{1/2}$.

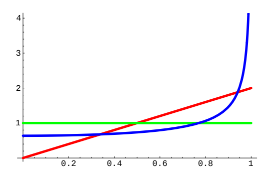


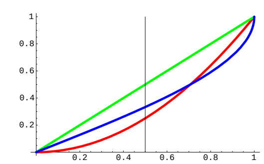
Answer Nr. 3: if the midpoint of the intersection with the disc is located in the disc of radius 1/2 with area $\pi/4$, then the chord is longer than $\sqrt{3}$. The probability is $\boxed{1/4}$.



The paradox comes from the choice of the probability density function $f(x,y)$. In each case, there is a distribution function $f(x,y)$ which is radially symmetric.
The constant distribution $\boxed{f(x,y) = 1/\pi}$ is obtained when we throw the center of the line into the disc. The disc A_r of radius r has probability r^2/π. The density in the r direction is $2r$.
The distribution $\boxed{f(x,y) = 1/r}$ is obtained when throwing parallel lines. This will put more weight to center. The disc A_r of radius r has probability of A_r is bigger than the area of A_r. The density in the r direction is constant equal to 1.
Lets compute the distribution when we rotate the line around a point at the boundary. We hit a disc A_r of radius r with probability $F(r) = \arcsin(r)2/\pi$. The density in the radial direction is $\boxed{f(r) = 2/(\pi\sqrt{(1-r^2)})}$.

COMPARISON OF THE DENSITY FUNCTIONS. A plot of the radial distribution functions $f(r)$ as well as $F(r)$, the probability of A_r shows why we get different results for $F(1/2)$.





What happens if we **really** do an experiment and throw randomly lines onto a disc? The outcome of the experiment will depend on how the experiment will be performed. If we would do the experiment by hand, we would probably try to throw the center of the stick into the middle of the disc. Since we would aim to the center, the distribution would probably be different from any of the three solutions discussed above.

STATISTICS. If $f(x,y)$ is a probability distribution on R: $f(x,y) \geq 0$, $\iint_{\mathbf{R}} f(x,y) \; dA = 1$, then $E[X] = \int X(x,y)f(x,y) \; dA$ is called the **expectation** of X, $Var(X) = E[(X - E[X])^2]$ is called the **variance** and $\sigma(X) = \sqrt{(Var[X])}$ the **standard deviation**.