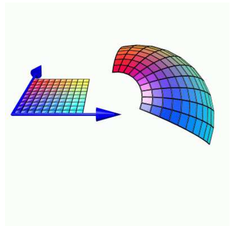
PARAMETRIC SURFACES — Maths21a, O. Knill

PARAMETRIC SURFACES. The image of a map

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

defines a **surface**. r is called the **parametrisation** of the surface. The surface is defined by three functions $x(u,v), y(u,v), z(u,v)$ each is a function of two variables. If we fix one of the variables, say $v = v_0$, then $u \mapsto \vec{r}(u,v_0)$ is a curve on the surface. Similarly, if we fix $u = u_0$, then $v \mapsto \vec{r}(u_0,v)$ is a curve on the surface.

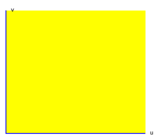


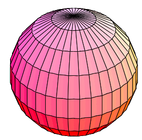
TWO WAYS TO REPRESENT A SURFACE.

- I) Solutions of an equation $g(x,y,z) = 0$.
- II) Parameterizations as image of $\vec{r}(u,v)$.

EXAMPLE GRAPH. The graph of a function $f(x,y)$ can be parametrized as $(u,v) \mapsto \vec{r}(u,v) = (u,v,f(u,v))$ and can also be written as a level surface: $\{g(x,y,z) = z - f(x,y) = 0\}$.

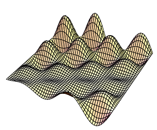
EXAMPLE SPHERE. The sphere is the set of (x,y,z) for which $g(x,y,z) = x^2+y^2+z^2 = 1$. The sphere is also the image of the parametrisation $\vec{r}(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$. (The upper half sphere is the graph of the function $f(x,y) = \sqrt{1-x^2-y^2}$. We could also write the sphere as a graph of $\rho(\theta,\phi) = r$ in spherical coordinates).





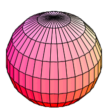
NOTE. There are surfaces, which can not be represented as an image of a single parameterization $\vec{r}$ and where different patches are needed. To do so, Mathematicians introduced the notion of a "manifold".

EXAMPLES. of parameterized surface $(u,v) \mapsto \vec{r}(u,v)$



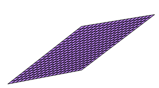
Graphs

$$\begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$$



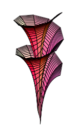
Sphere

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix}$$



Plane through points P, Q, R

$$\begin{matrix} P+ \\ u(Q-P)+ \\ v(R-P) \end{matrix}$$

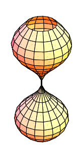


Dini surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v)+\log(\tan(\frac{v}{2}))+\frac{u}{5} \end{bmatrix}$$

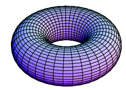
SURFACE OF REVOLUTION.

When spinning a graph $y = f(x)$ around the x-axis, we obtain a **surface of revolution**. Keeping $u = x$ as one of the parameters and v as the angle of rotation and $f(u)$ as the radius, we get $x(u,v) = u, y(u,v) = f(u)\cos(v), z(u,v) = f(u)\sin(v)$. Therefore, $\vec{r}(u,v) = (u, f(u)\cos(v), f(u)\sin(v))$.
For example, for $f(x) = x$, we obtain a cone, for $f(x) = x^2$, we obtain a paraboloid, the surface with $f(x) = 1+\sin(x)$ is plottet.



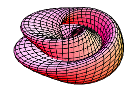
GRID CURVES. If we keep u constant, then $v \mapsto \vec{r}(u,v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u,v)$ is a curve on the surface. These curves are called **grid curves**. If you plot a surface with a computer, the pictures usually show these grid curves.

MORE EXAMPLES. Parameterized surfaces $(u,v) \mapsto \vec{r}(u,v)$



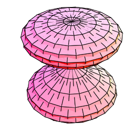
Torus

This is a homework



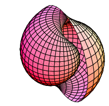
Klein bottle

$$\begin{bmatrix} (\frac{3}{2}+\cos(u)\sin(2v)-\sin(u)\sin(4v))\cos(2u) \\ (\frac{3}{2}+\cos(u)\sin(2v)-\sin(u)\sin(4v))\sin(2u) \\ \frac{3}{4}\sin(u)\sin(2v)+\cos(u)\sin(4v) \end{bmatrix}$$



Eight surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v)+\log(\tan(\frac{v}{2}))+\frac{u}{5} \end{bmatrix}$$

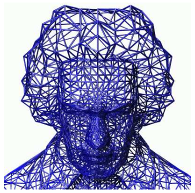


Snail surface

$$\begin{bmatrix} \cos(u)\sin(2v) \\ \sin(u)\sin(2v) \\ \sin(v) \end{bmatrix}$$

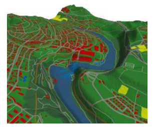
WHERE DO SURFACES OCCUR?

Computer graphics. i.e. modeling of faces, terrains, cars, spaceships etc. The map $\vec{r}$ is called a **u-v map**. Such maps are used to apply textures to three-dimensional models. There are software utilities, which allows an artist to draw directly onto the R domain. The software then applies the map automatically onto the surface. You can paint like this on surfaces in space.



Level surfaces. A level surface of a function of three variables is the set $g(x,y,z) = c$, where c is a constant. Examples are **isotherms**, surfaces of constant temperature in the ocean, **isobars**, surfaces of constant pressure in the atmosphere.

Graphs. For example the height function $h(x,y)$ which gives the height $h(x,y)$ at a point (x,y).



Geometric Intuition. Intuition for higher dimensional surfaces or "manifolds" is often obtained from 2-dimensional surfaces in three dimensional space. Higher dimensional surfaces appear everywhere. Our universe for example is modeled as a space time manifold in general relativity. The planetary motion in our the solar system is modeled by a motion on a $9 \cdot 6 - 10 = 44$ dimensional surface. Even so, we can not draw these objects, calculus allows to work with such objects as in two or three dimensions.

D-branes. In super-string theory physicists consider surfaces called **Dp-branes**. The letter D stands for "Dirichlet". $D1$-branes are called **D-strings**. $D2$-branes are two dimensional surfaces.

Art. Last but not least, there is the artistic or esthetic aspect. Examples are paintings by Fomenko, a Russian topologist. An other example is the surface "topological" at the entrance of the science center visible on the photo. You pass this surface every day.