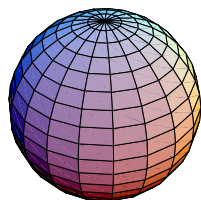


VOLUME SPHERE/TORUS

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We first calculate the volume of a sphere of radius R in different ways. Then we show how to calculate the volume of the torus in three different ways. The page serves more as an illustration for the variety of tools which are available. We did not cover all the material (yet) to understand all of the methods.



SEVEN WAYS TO COMPUTE THE VOLUME OF THE SPHERE

1) IN RECTANGULAR COORDINATES. The volume is

$$V = \int \int \int_R dx dy dz = \int_{-L}^L \int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} \int_{-\sqrt{L^2-x^2-y^2}}^{\sqrt{L^2-x^2-y^2}} dz dy dx$$

After computing the most inner integral, we have $V = \int_{-L}^L \int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} 2(L^2-x^2-y^2)^{1/2} dy dx$. Define $a^2 = L^2 - x^2$ and use the integral in the box below:

$$V = \pi \int_{-L}^L (L^2 - x^2) dx = 2\pi L^3 - 2\pi L^3/3 = 4\pi L^3/3.$$

Substitution $\frac{x}{a} = \sin(u)$, $dx = a \cos(u) du$ gives:

$$\left[\int_{-a}^a \sqrt{a^2 - x^2} dx \right] = \int_{-\pi/2}^{\pi/2} a \sqrt{1 - \sin^2(u)} a \cos(u) du = a^2 \int_{-\pi/2}^{\pi/2} \cos^2(u) du = \boxed{a^2 \pi / 2}.$$

2) IN CYLINDRICAL COORDINATES. At height z , we parameterize a disc of radius $L^2 - z^2$, so that the integral is

$$\int_{-L}^L \int_0^{\sqrt{L^2-z^2}} \int_0^{2\pi} r dr d\theta dz = \int_{-L}^L (L^2 - z^2) 2\pi dz = 4\pi L^3 - 2\pi L^3/3 = 4\pi L^3/3.$$

3) IN SPHERICAL COORDINATES.

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 \sin(\phi) d\phi d\theta d\rho = 2\pi \int_0^R \rho^2 d\rho \int_0^\pi \sin(\phi) d\phi = 4\pi L^3/3.$$

4) WITH CAVALIERI. Cavalieri cuts the hemisphere at height z to obtain a disc of radius $\sqrt{L^2 - z^2}$ with area $\pi(L^2 - z^2)$. He looked now at the complement of a cone of height L and radius L which when cut at height z gives a ring of outer radius L and inner radius z . The ring has area $\pi(L^2 - z^2)$. Cavalieri concludes (Cavalieri principle) that the volume of that body is the same as the volume of the hemisphere. Since the difference of the volume of the cylinder and the cone which is $\pi L^3 - \pi L^3/3$ the hemisphere has the volume $2\pi L^3/3$ and the sphere has volume $4\pi L^3/3$.

5) CAS. Integrate[1, {x, -L, L}, {y, -Sqrt[L^2 - x^2], Sqrt[L^2 - x^2]}, {z, -Sqrt[L^2 - x^2 - y^2], Sqrt[L^2 - x^2 - y^2]}]

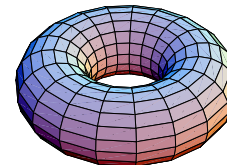
6) IN LAS VEGAS. The Monte Carlo Method is to shoot randomly onto the cube $[-L, L] \times [-L, L] \times [-L, L]$ and see how many times we hit the sphere. Here an experiment with Mathematica:

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R := (2Random[] - 1); k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k + +], {10000}]; k/10000
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Assume, we hit 5277 of 10000 the measured fraction of the volume of the sphere with the volume of the cube 8 is 0.5277. The volume of 1/8'th of the sphere is $\pi/6 = 0.524$

7) USING GAUSS THEOREM (see later) The vector field $F(x, y, z) = (x, y, z)$ has divergence 3 Gauss theorem tells that $3V$ is the flux of the vector field through the surface which is L times the surface area $4\pi L^2$. Therefore, $V = 4\pi L^3/3$

FIVE WAYS TO COMPUTE THE VOLUME OF THE TORUS



1) WITH TORAL COORDINATES.

$T(r, \theta, \phi) = (x, y, z) = ((b + r \cos(\phi)) \cos(\theta), (b + r \cos(\phi)) \sin(\theta), r \sin(\phi))$ parameterizes the torus.

The Jacobean is $\det(T') = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r(b + r \cos(\phi))$. The torus is the image of the cube $[0, a] \times [0, 2\pi] \times [0, 2\pi]$ under the map T . The change of variables formula gives

$$\int_0^a \int_0^{2\pi} \int_0^{2\pi} r(b + r \cos(\phi)) d\phi d\theta dr = (2\pi)(2\pi) \int_0^a br dr = 2\pi^2 a^2 b$$

2) USING CYLINDRICAL COORDINATES.

If we fix the z coordinate, we obtain an annulus with inner radius $b - \sqrt{a^2 - z^2}$ and outer radius $b + \sqrt{a^2 - z^2}$. This annulus has the area $\pi(b + \sqrt{a^2 - z^2})^2 - \pi(b - \sqrt{a^2 - z^2})^2$. Therefore, the volume is $4\pi b \int_{-a}^a \sqrt{a^2 - z^2} dz = 4\pi b(\pi a/2) = 2\pi^2 a^2 b$.

3) USING PAPPUS CENTROID THEOREM. "The volume of a solid of revolution generated by the revolution of a region S in the $x - z$ plane around the z axis is equal to the product of the area of S and the arc length $2\pi b$ of the circle on which the center of S moves".

In the case of the torus, the length of the curve is $2\pi b$. The area of the lamina is $A = \pi a^2$. Therefore, the volume is $2\pi^2 a^2 b$.

PROOF OF THE CENTROID THEOREM. We use a coordinate change transformation. In Polar coordinates, the lamina S with center of mass (b, c) is parametrized by r and z . Introduce new coordinates $T(u, v) = (u + b, v + c) = (r, z)$ so that $(0, 0)$ is the center of mass in the new coordinates. The Jacobean of this coordinate change is 1. The volume of the solid of revolution is $V = (2\pi) \int_S r dr dz = (2\pi) \int_R (u + b) dudv = 2\pi b \int \int b dudv = 2\pi b A$, where we used that $\int \int_R u dudv = 0$ because $(u, v) = (0, 0) = (\int \int_R u dudv/A, \int \int_R v dudv/A)$ is the center of mass of R .

4) MONTE CARLO AGAIN. Lets assume $b = 2$ and $a = 1$. If (x, y, z) is a random point in $[-3, 3] \times [-3, 3] \times [-1, 1]$ then $(r - 2)^2 + z^2 \leq 1$ is the condition to be in the torus, where $r^2 = x^2 + y^2$. Assume, we hit 5484 of 100000 the measured fraction of the volume 72 of the box we estimate $72 * 5484 / 100000 = 3.94848$ for the actual volume $4\pi^2 = 39.4784$.

5) CAS. Integrate[r, {r, 1, 3}, {theta, 0, 2Pi}, {z, -Sqrt[1 - (r - 2)^2], Sqrt[1 - (r - 2)^2]}]