VECTORS, DOT PRODUCT

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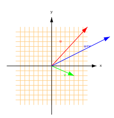
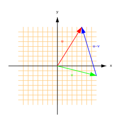
VECTORS. Two points $P_1 = (x_1, y_1)$, $Q = P_2 = (x_2, y_2)$ in the plane determine a **vector** $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. It points from P_1 to P_2 and we can write $P_1 + \vec{v} = P_2$.
COMPONENTS. Points P in space are in one to one correspondence to vectors pointing from 0 to P. The numbers $\vec{v}_i$ in a vector $\vec{v} = (v_1, v_2)$ are also called **components** or of the vector.
REMARKS: vectors can be drawn **everywhere** in the plane. If a vector starts at 0, then the vector $\vec{v} = \langle v_1, v_2 \rangle$ points to the point $\langle v_1, v_2 \rangle$. That's is why one can identify points $P = (a, b)$ with vectors $\vec{v} = \langle a, b \rangle$. Two vectors which can be translated into each other are considered **equal**. In three dimensions, vectors have three components. In some Encyclopedias vectors are defined as objects which have "both magnitude and direction". This is unprecise and strictly speaking incorrect because the **zero vector** is a vector with no direction.

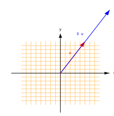
ADDITION SUBTRACTION, SCALAR MULTIPLICATION.



$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$



$\vec{u} - \vec{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle = \langle u_1 - v_1, u_2 - v_2 \rangle$

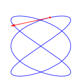


$\lambda \vec{u} = \lambda \langle u_1, u_2 \rangle = \langle \lambda u_1, \lambda u_2 \rangle$

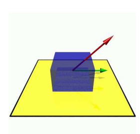
BASIS VECTORS. The vectors $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ are called **standard basis vectors** in the plane. In space, one has the basis vectors $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$.
Every vector $\vec{v} = (v_1, v_2)$ in the plane can be written as a sum of standard basis vectors: $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$. Every vector $\vec{v} = (v_1, v_2, v_3)$ in space can be written as $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

WHERE DO VECTORS OCCUR? Here are some examples:

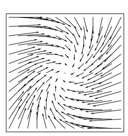
Velocity: if $(f(t), g(t))$ is a point in the plane which depends on time t, then $\vec{v} = \langle f'(t), g'(t) \rangle$ is the **velocity vector** at the point $(f(t), g(t))$.



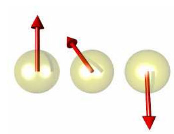
Forces: Some problems in statics involve the determination of a forces acting on objects. Forces are represented as vectors



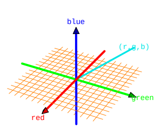
Fields: electromagnetic or gravitational fields or velocity fields in fluids are described with vectors.



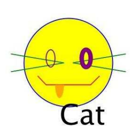
Qbits: in quantum computation, rather than working with bits, one deals with **qbits**, which are vectors.



Color can be written as a vector $\vec{v} = (r, g, b)$, where r is red, g is green and b is blue. An other coordinate system is $\vec{v} = (c, m, y) = (1 - r, 1 - g, 1 - b)$, where c is cyan, m is magenta and y is yellow.



SVG. Scalable Vector Graphics is an emerging standard for describing two-dimensional graphics in XML.



VECTOR OPERATIONS: The addition and scalar multiplication of vectors satisfy "obvious" properties. There is no need to memorize them. We write $*$ here for multiplication with a scalar but usually, the multiplication sign is left out.

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$	commutativity
$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$	additive associativity
$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{0}$	null vector
$r * (s * \vec{v}) = (r * s) * \vec{v}$	scalar associativity
$(r + s)\vec{v} = \vec{v}(r + s)$	distributivity in scalar
$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$	distributivity in vector
$1 * \vec{v} = \vec{v}$	the one element

LENGTH. The length $|\vec{v}|$ of $\vec{v}$ is the distance from the beginning to the end of the vector.

EXAMPLES. 1) If $\vec{v} = (3, 4)$, then $|\vec{v}| = \sqrt{25} = 5$. 2) $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$, $|\vec{0}| = 0$.

UNIT VECTOR. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.
EXAMPLE: If $\vec{v} = (3, 4)$, then $\vec{v} = (3/5, 4/5)$ is a unit vector, $\vec{i}, \vec{j}, \vec{k}$ are unit vectors.

PARALLEL VECTORS. Two vectors $\vec{v}$ and $\vec{w}$ are called **parallel**, if $\vec{v} = r\vec{w}$ with some constant r.

DOT PRODUCT. The **dot product** of two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is defined as

$$\boxed{\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

Remark: in science, other notations are used: $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$ (mathematics) $< \vec{v} | \vec{w} >$ (quantum mechanics) $v_i w^i$ (Einstein notation) $g_{ij} v^i w^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

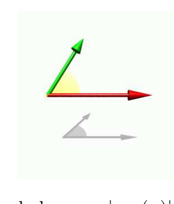
LENGTH. Using the dot product one can express the length of $\vec{v}$ as $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.

CHALLENGE. Express the dot product in terms of the length alone.

SOLUTION: $(\vec{v} + \vec{w}, \vec{v} + \vec{w}) = (\vec{v}, \vec{v}) + (\vec{w}, \vec{w}) + 2(\vec{v}, \vec{w})$ can be solved for $(\vec{v}, \vec{w})$.

ANGLE. Because $|\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}, \vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v}, \vec{w})$ is by the **cos-theorem** (which is easy to prove) equal to $|\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha)$, where α is the angle between the vectors $\vec{v}$ and $\vec{w}$, we get the important formula

$$\boxed{\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)}$$

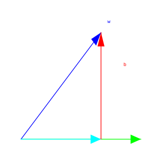


CAUCHY-SCHWARZ INEQUALITY: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$ follows from that formula because $|\cos(\alpha)| \leq 1$.
TRIANGLE INEQUALITY: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ follows from $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1, 4, 3)$ and $(-1, 2, 3)$.
ANSWER: $\cos(\alpha) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$. So that $\alpha = \arccos(0.839..) \sim 33°$.

ORTHOGONAL VECTORS. Two vectors are called **orthogonal** (= **perpendicular**)if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. EXAMPLE: $\vec{v} = (2, 3)$ is orthogonal to $\vec{w} = (-3, 2)$.
PYTHAGORAS. If $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$. Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$.

PROJECTION. The vector $\boxed{\vec{a} = \text{proj}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w}/|\vec{w}|^2)}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$.
The **scalar projection** is defined as $\boxed{\text{comp}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \vec{w})/|\vec{w}|}$. (Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$.) The vector $\vec{b} = \vec{v} - \vec{a}$ is called the **component** of $\vec{v}$ orthogonal to the $\vec{w}$-direction.



EXAMPLE. $\vec{v} = (0, -1, 1)$, $\vec{w} = (1, -1, 0)$, $\text{proj}_{\vec{w}}(\vec{v}) = (1/2, -1/2, 0)$, $\text{comp}_{\vec{w}}(\vec{v}) = 1/\sqrt{2}$.