Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

1) T F If $\vec{u}+\vec{v}+\vec{w}=\vec{0}$ then $\vec{u}\cdot(\vec{v}\times\vec{w})=0$.

Solution:
If $\vec{u}+\vec{v}+\vec{w}=\vec{0}$, then $\vec{u}$ and $\vec{v}$ and $\vec{w}$ are in the same plane and the triple scalar product is zero, because it is the volume of the parallel epiped spanned by the three vectors.

2) T F $\int_0^5\int_0^\pi r\,d\theta\,dr$ is half the area of a disc radius 5 in the plane.

Solution:
This is indeed the formula of the area in polar coordinates.

3) T F If a vector field $\vec{F}(x,y)$ satisfies $\mathrm{curl}(F)(x,y)=0$ for all points (x,y) in the plane, then $\vec{F}$ is conservative.

Solution:
True. We have derived this from Greens theorem.

4) T F If the acceleration of a parameterized curve $\vec{r}(t)=(x(t),y(t),z(t))$ is zero then the curve $\vec{r}(t)$ is a line.

Solution:
If we integrate, then $\vec{r}'(t)$ is a constant vector $\vec{v}$. Then $\vec{r}(t)=\vec{r}(0)+t\vec{v}$.

5) T F A circle of radius $1/2$ has a smaller curvature than a circle of radius 1.

Solution:
The curvature of a circle of radius r is equal to $1/r$.

6) T F The curve $\vec{r}(t)=(-\sin(t),\cos(t))$ for $t\in[0,\pi]$ is half a circle.

Solution:
True. Indeed, one can check that $\sin^2(t)+\cos^2(t)=1$.

7) T F The function $u(t,x) = \sin(x+t)$ is a solution of the partial differential equation $u_{tx} + u = 0$

Solution:
$u_x = -\cos(x+t)$, $u_t = -\cos(x+t)$ and $u_{xt} + u = 0$.

8) T F The length of a curve $\vec{r}(t)$ in space parameterized on $a \leq t \leq b$ is the value of the integral $\int_1^2 |\vec{T}\,'(t)|\,dt$, where $\vec{T}(t)$ is the unit tangent vector.

Solution:
The correct solution is $\int_1^2 |\vec{r}\,'(t)|\,dt$.

9) T F Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then the gradient of g at (x_0, y_0) is parallel to the gradient of f at (x_0, y_0).

Solution:
This paraphrases indeed part of the Lagrange equations.

10) T F At a point which is not a critical point, the directional derivative $D_{\vec{v}} f(x_0, y_0, z_0)$ can take both the negative and the positive sign.

Solution:
For $\vec{v} = \nabla f$, the directional derivative is positive. For $\vec{v} = -\nabla f$, the directional derivative is negative.

11) T F If a vector field $\vec{F}(x,y)$ is a gradient field, we always can find a curve C for which the line integral $\int_C \vec{F} \cdot d\vec{r}$ is positive.

Solution:
While there does not exist a closed curve with this property, there are many curves, if the field $\vec{F}$ is not zero. Just move a bit into a direction of $\vec{F}$ at a point, where $\vec{F}$ is not zero.

12) T F If C is a closed level curve of a function $f(x,y)$ and $\vec{F} = (f_x, f_y)$ is the gradient field of f, then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:
The gradient field is perpendicular to the level curves.

13) T F The divergence of a gradient vector field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is always zero.

Solution:
Just take a simple example like $f(x,y,z) = x^2$, where $\mathrm{div}(\mathrm{grad}(f) = 2$. Actually, $\mathrm{div}(\mathrm{grad}(f) = \Delta f$ is the Laplacian of f.

14) T F The line integral of the vector field $\vec{F}(x,y,z) = \langle x^2, y^2, z^2 \rangle$ along a line segment from $(0,0,0)$ to $(1,1,1)$ is 1.

Solution:
By the fundamental theorem of line integrals, we can take the difference of the potential $f(x,y,z) = x^3/3 + y^3/3 + z^3/3$, which is $1/3 + 1/3 + 1/3$.

15) T F If $\vec{F}(x,y) = (x^2 - y, x)$ and $C : \vec{r}(t) = \langle \sqrt{\cos(t)}, \sqrt{\sin(t)} \rangle$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C F \cdot\, ds$ is twice the area of R.

Solution:
This is a direct consequence of Green's theorem and the fact that the two-dimensional curl $Q_x - P_y$ of $\vec{F} = \langle P, Q \rangle$ is equal to 2.

16) T F The flux of the vector field $\vec{F}(x,y,z) = \langle 0, y, 0 \rangle$ through the boundary S of a solid sphere E is equal to the volume the sphere.

Solution:
It is the **volume** of the solid torus.

17) T F The quadratic surface $-x^2 + y^2 + z^2 = 5$ is a one-sheeted hyperboloid.

Solution:
The traces are a circle and hyperboloids.

18) T F If $\vec{F}$ is a vector field in space and S is the boundary of a sphere then the flux of $\vec{\mathrm{curl}}(\vec{F})$ through S is 0.

Solution:
This is true by Stokes theorem.

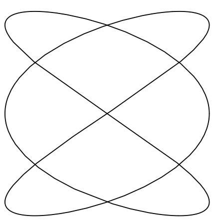
19) [T] [F] If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) and S is a torus surface, then the flux of $\vec{F}$ through S is zero.

Solution:
This is a consequence of the divergence theorem.

20) [T] [F] In spherical coordinates, the equation $\rho\cos(\phi) = \rho\cos(\theta)\sin(\phi)$ defines a plane.

Solution:
True. It is the plane $z = x$.

Problem 2) (10 points)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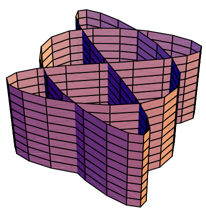
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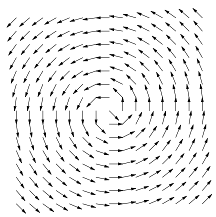
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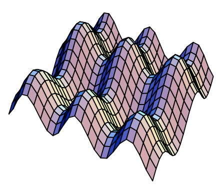
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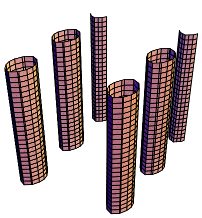
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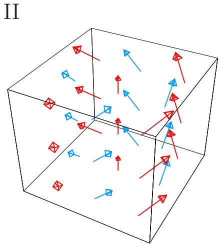
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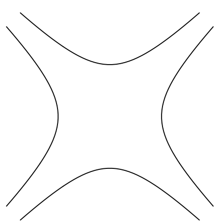
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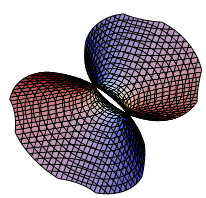

VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$x^2 - y^2 + z^2 = 1$
	$\vec{r}(t) = \langle \cos(3t), \sin(2t) \rangle$
	$z = f(x,y) = \cos(3x) + \sin(2y)$
	$\vec{F}(x,y) = \langle -y/\sqrt{x^2+y^2}, x/\sqrt{x^2+y^2} \rangle$
	$\cos(3x) + \sin(2y) = 1$
	$\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$
	$\vec{r}(u,v) = \langle \cos(3u), \sin(2u), v \rangle$
	$\{(x,y) \in \mathbf{R}^2 \mid \lvert x^2 - y^2 \rvert = 1\}$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
VIII	$x^2 - y^2 + z^2 = 1$
I	$\vec{r}(t) = \langle \cos(3t), \sin(2t) \rangle$
IV	$z = f(x,y) = \cos(3x) + \sin(2y)$
III	$\vec{F}(x,y) = \langle -y/\sqrt{x^2+y^2}, x/\sqrt{x^2+y^2} \rangle$
V	$\cos(3x) + \sin(2y) = 1$
VI	$\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$
II	$\vec{r}(u,v) = \langle \cos(3u), \sin(2u), v \rangle$
VII	$\{(x,y) \in \mathbf{R}^2 \mid \lvert x^2 - y^2 \rvert = 1\}$

Furthermore, fill in the peoples names, Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

Formula	Name of the theorem
$\int_C \vec{F} \cdot d\vec{r} = \int\int_S \text{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x,y) = f_{yx}(x,y)$	
$\int_C \vec{F} \cdot dr = \int\int_R \text{curl}(\vec{F})\ dxdy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int\int_S F \cdot dS = \int\int\int_E \text{div}(F)\ dV$	
$\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx$	

Solution:

Formula	Name of the theorem
$\int_C \vec{F} \cdot \vec{dr} = \int \int_S \text{curl}(\vec{F}) \cdot dS$	Stokes
$f_{xy}(x,y) = f_{yx}(x,y)$	Clairot
$\int_C \vec{F} \cdot dr = \int \int_R \text{curl}(\vec{F})\ dxdy$	Green
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$	Fundamental theorem of line integrals
$\int \int_S F \cdot dS = \int \int \int_E \text{div}(F)\ dV$	Gauss
$\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx$	Fubini

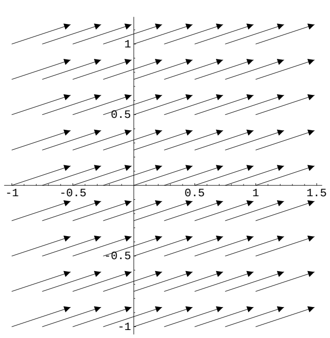
Problem 3) (10 points)

In this problem, vector fields $\vec{F}$ are written as $\vec{F} = \langle P, Q \rangle$. We use abbreviations $\text{curl}(F) = Q_x - P_y$. When stating $\text{curl}(F) = 0$, we mean that $\text{curl}(F)(x,y) = 0$ vanishes for **all** (x,y). Similarly, we say $\text{div}(F)$ if $\text{div}(F)(x,y) = P_x(x,y) + Q_y(x,y) = 0$ for all x, y.

Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV and mark also the places, indicating the vanishing of $\text{curl}(F)$.

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{div}(F) = 0$
$\vec{F}(x,y) = \langle 1, x \rangle$						
$\vec{F}(x,y) = \langle 3y, -3x \rangle$						
$\vec{F}(x,y) = \langle 7, 2 \rangle$						
$\vec{F}(x,y) = \langle x, y \rangle$						

I



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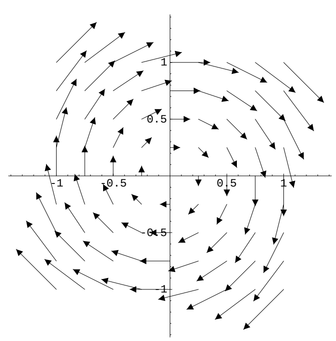

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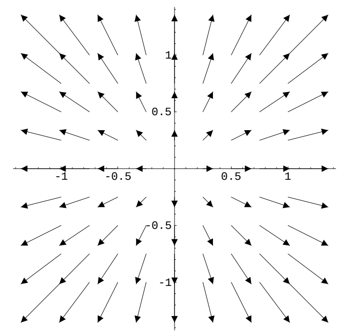

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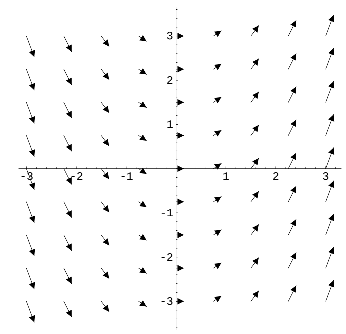


Solution:

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{div}(F) = 0$
$\mathbf{F}(x,y) = \langle y, 0 \rangle$				X		X
$\mathbf{F}(x,y) = \langle y, -x \rangle$		X				X
$\mathbf{F}(x,y) = \langle 0, 5 \rangle$	X				X	X
$\mathbf{F}(x,y) = \langle -x, -y \rangle$			X		X	

Problem 4) (10 points)

a) (5 points) What is the area of the triangle A, B, P, where $A = (1,1,1), B = (1,2,3)$ and $P = (3,2,4)$?

b) (5 points) Find the distance between the point the point P and the line L passing through the points A with B.

Solution:
a) The area is half of the cross product of $\vec{AB}$ and $\vec{AP}$ which is $(0,1,2) \times (2,1,3)$ which is $|(1,4,-2)|$ which is $\sqrt{21}$. The triangle has the area $\sqrt{21}/2$.
b) The distance formula is $|\vec{AB} \times \vec{AP}|/|\vec{AB}| = |(1,4,-2)|/|(2,1,3)| = \frac{\sqrt{21}}{5}$.

Problem 5) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x,y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2} .$$

There is a lake in that area as you can see in the photo.

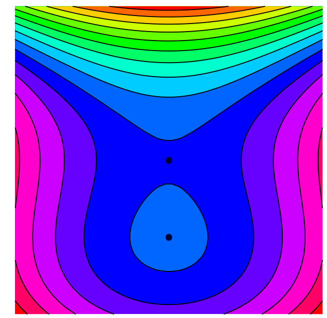
a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.

b) (3 points) For any pair of two different critical points A, B found in $a)$ let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \ d\vec{r}$.

Photo of the lake in the Swiss alps near the Simplon mountain pass.

Solution:
a) The gradient is $\nabla f(x,y) = (x-1, -y-y^2)$. This gradient vanishes if $x = 1$ and $y = -1$ or $y = 0$. So, there are two critical points $(1,-1),(1,0)$. The Hessian matrix is $H(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -y-y^2 \end{bmatrix}$.

point	discriminant	f_{xx}	nature
$(1,-1)$	D= 1	1	min
$(1,0)$	D= -1	1	saddle



b) By the fundamental theorem of line integrals, the line integral between the two points is the difference of the potentials which is $f(1,-1) - f(1,0) = (-1+1/3-1/2+1/2) - (-1+1/2) = -1/6$. Also the answer $1/6$ is correct of course, since we did not specify the direction.

Problem 6) (10 points)

a) (4 points) Find the linearization $L(x,y,z)$ of $f(x,y,z) = 2 + z - \sin(-x-3y)$ at the point $P = (0,\pi,2)$.

b) (4 points) Find the equation of the tangent plane at that point $P = (0,\pi,2)$.

c) (2 points) Estimate $f(0.001,\pi,2.02)$ using the linearization.

Solution:
a) $\nabla f(x,y,z) = (\cos(-x+3y), 3\cos(-x+3y), 1)$. At the point $(0,\pi,2)$, we have $\nabla f(0,\pi,2) = (-1,-3,1)$. We have $f(0,\pi,2) = 4$. The linearization is

$$L(x,y,z) = 4 + (-1,-3,1) \cdot (x, y-\pi, z-2) = 2 + 3\pi - x - 3y + z \ .$$

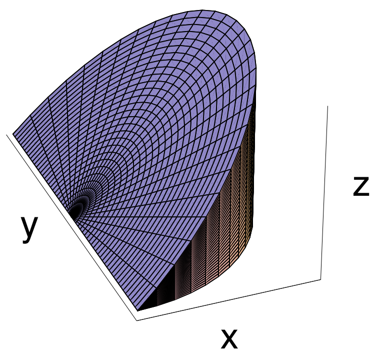
b) From the gradient $\nabla f(0,\pi,2) = (a,b,d) = (-1,-3,1)$, we know that the equation is $ax+by+cz = d$ which is $-x-3y+z = d$. We get the constant d by pluggin in the point. The equation is

$$-x - 3y + z = -3\pi + 2 \ .$$

c) Evaluate L at the point: $L(0.001, \pi, 2.02) = 4 + (-1,-3,1) \cdot (0.001, 0, 0.02) = 4.019$.

Problem 7) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the solid cylinder $x^2 + y^2 \leq 9$.



Solution:
Use cylindrical coordinates and note that the wedge is positive on the right half plane so that we have to integrate overt right half of the unit disc:

$$\int_0^3 \int_{-\pi/2}^{\pi/2} r^2 \cos(\theta) \ d\theta dr = 54/3 = 18 \ .$$

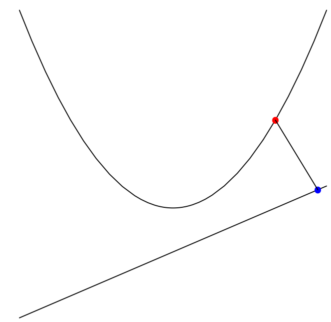
Note that spherical coordinates would not be good, since the bounds for the angle ϕ depend on the angle θ. The solution is $\boxed{18}$.

Problem 8) (10 points)

The distance from a point (x,y) to the line $y = x$ in the plane is given by $f(x,y) = (y-x)/\sqrt{2}$. Use the Lagrange method to find the point (x,y) on the parabola

$$g(x,y) = x^2 - y = -2$$

which is closest to the line.



Solution:
We have to extremize the function $f(x,y)$ under the constraint $g(x,y) = -2$. The Lagrange equations are

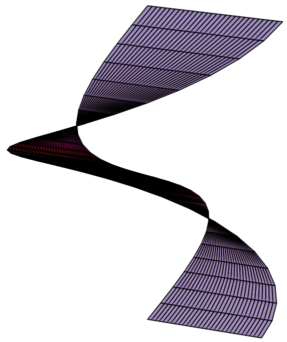
$$\begin{aligned} -1/\sqrt{2} &= \lambda 2x & (1) \\ 1/\sqrt{2} &= -\lambda & (2) \\ x^2 - y &= -2 & (3) \end{aligned}$$

We have $x = 1/2$ and $y = 9/4$. The point $(1/2, 9/4)$ on the parabola is closest to the line.

Problem 9) (10 points)

a) (5 points) A ribbon of a girl is modeled as a surface S which is parameterized by $\vec{r}(t,s) = (s\cos(t), \sin(t), t)$, where $t \in [0, 2\pi]$ and $s \in [0, 1]$. Find the surface area of this ribbon S.

b) (5 points) Part of the boundary of the ribbon is obtained when fixing $s = 1$. It is a curve in space. Find the arc length of this curve $\vec{r}(t)$, parametrized from $t = 0$ to 2π.



Painting: "Young Girl with Blue Ribbon" by the French painter Jean-Baptiste Greuze (1725-1805)

Solution:
a) We have to compute the integral $\int_0^{2\pi} \int_0^1 |r_s \times r_t| \; dsdt$. We have $r_s = (\cos(t), 0, 0)$ and $r_t = (-s\sin(t), \cos(t), 1)$. Now $r_s \times r_t = \cos(t)(0, -1, \cos(t))$ and $|r_s \times r_t| = |\cos(t)|\sqrt{1 + \cos^2(t)}$. The integral

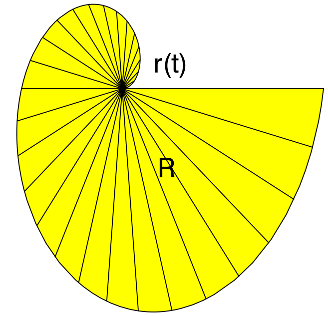
$$2\int_{-\pi/2}^{\pi/2} \cos(t)\sqrt{1 + \cos^2(t)} \; dt$$

can be solved using integration by parts and some trigonometric identities to get $2 + \pi$. (We give full credit for the correct integral already).
b) For $s = 1$, we have a helix $\vec{r}(t) = (-\sin(t), \cos(t), t)$ which has the speed $|r'(t)| = \sqrt{2}$ and the arc length is $\int_0^{2\pi} \sqrt{2} \; dt = 2\pi\sqrt{2}$.

Problem 10) (10 points)

A region R in the xy-plane is given in polar coordinates by $0 \leq r(\theta) \leq \theta$ for $\theta \in [0, 2\pi]$. You see the region in the picture to the right. Its boundary is called the **Archimedes spiral**. It can be found on the tomb of Jacob Bernoulli. Evaluate the double integral

$$\iint_R \frac{e^{-x^2-y^2}}{\sqrt{x^2+y^2}} \; dx \; dy \; .$$



Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = drd\theta$ does not work. The integral $\int_0^{2\pi} \int_0^{\theta} e^{-r^2} \; drd\theta$ can not be solved. We have to change the order of integration:

$$\int_0^{2\pi} \int_0^r e^{-r^2} \; d\theta \; dr$$

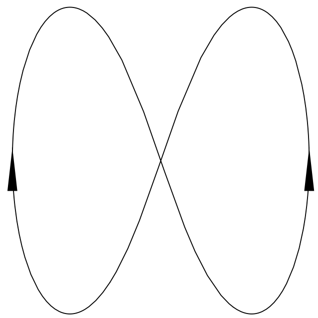
Evaluating the inner integral gives $\int_0^{2\pi} e^{-r^2} r \; dr = \boxed{(1 - e^{-4\pi^2})2}$.

Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x,y) = \langle -y + y\sin(xy), x\sin(xy) \rangle$ along the boundary C of the figure eight curve

$$\vec{r}(t) = \langle \frac{\cos(t)}{(1+\sin(t)^2)}, \frac{\sin(t)\cos(t)}{(1+\sin(t)^2)} \rangle \; ,$$

where $t \in [0, 2\pi]$. The orientation of the curve is indicated by the arrows.



Solution:
The curl is constant 2 so that by Green's theorem, the line integral is twice the area of R. We have two identical regions, but the orientation on one region is opposite to the orientation of the second region. The answer is 0. (In the exam, the left arrow had been oriented incorrectly so that we give full credit if Green was applied correctly and noted that the result has is related to an area).

Problem 12) (10 points)

Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = \langle -z + x^{(x^x)}, 5 + y^{(y^y)}, y + z^{(z^z)} \rangle$.
Let C be the curve given by the parameterization $\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$, for $0 \leq t \leq 2\pi$.
Compute the line integral of $\vec{F}$ along C.

Hint. You might want to consider a surface contained in the xz-plane which is enclosed by the curve.

Solution:
We use Stokes Theorem. The curl of the vector field is $\langle 1, -1, 0 \rangle$. The parameterization describes the circle $x^2 + z^2 = 1$, where $y = 0$. The curve starts at $(1,0,0)$ and rotates towards back towards $(0,0,1)$. By Stokes theorem, the line integral can be computed as the flux of curl(F) through the unit disk D in the xz plane which has the normal vector $\vec{r}_u \times \vec{r}_v = -\vec{j}$ and $\text{curl}(\vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = -1$. The flux is

$$\int \int_D -1 \; dx \; dz = -\pi \; .$$

The final answer is $\boxed{-\pi}$.

Problem 13) (10 points)

What is the flux of the vector field

$$\vec{F}(x,y,z) = \langle 3x + \cos(z^2 \sin(z)), x, \sin(y^3 + \cos(\sin(xy^3))) \rangle$$

through the boundary S of the solid cylinder $E = \{x^2 + y^2 \leq 1, \; 0 \leq z \leq 10 \; \}$. The surface of the cylinder is oriented so that the normal vector points outwards.

Solution:
We use the divergence theorem $\int \int_S F \cdot dS = \int \int \int \text{div}(F) \; dV$. The divergence is 3 so that the flux integral is 3 times the volume of the cylinder. The cylinder has volume $10 \cdot \pi$. The flux is therefore $\boxed{30\pi}$.