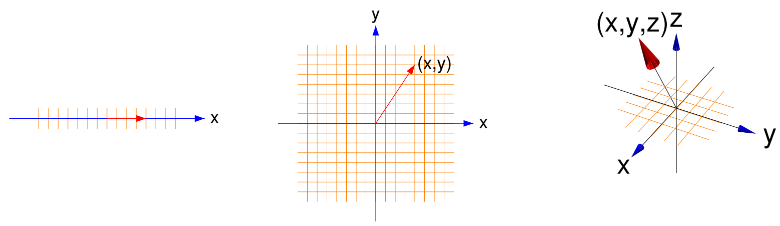
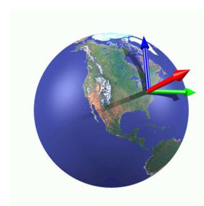
COORDINATES/DISTANCES — O. Knill, Maths21a, 2006

CARTESIAN COORDINATE SYSTEMS. A point on the line is labeled by a single coordinate x, a point in the **plane** is fixed by 2 coordinates (x, y) and a point in space is determined by three coordinates (x, y, z). Depending on which coordinates are positive, one can divide the line, the plane or the space into **half lines**, **quadrants** or **octants**.

1D space = line =2 half lines   2D space = plane=4 quadrants   3D space = space=8 octants

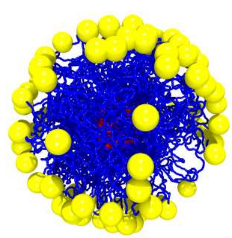


CHOICE OF COORDINATE SYSTEM.
Fixing the three coordinate axis determines a **coordinate system** in space. The choice of the convenient coordinate system depends on the situation. On earth for example the coordinate system is chosen so that the z-axis points "up" and is perpendicular to the ground. But these directions depend on the earth location of course.



EXAMPLE. Usually, we draw the coordinate system, such that the x,y coordinates are on the ground and the z-coordinate points "up". In two dimensions, on a sheet of paper, the x-coordinate usually is chosen to point "east" and the y-coordinate to point "north".

EXAMPLE. PHOTOGRAPHERS COORDINATE SYSTEM: In 3D graphics like computer games, virtual reality or ray tracing, it is custom to have the y-axis pointing up, the x-axis to the right and the z axis in front. This is the **"photographers coordinate system"**. If the photo is the x-y plane, then the depth is the z axis. Is the photographers coordinate system a left or right-handed coordinate system?



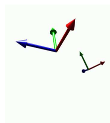
APPLICATION: Z-BUFFER. In computer graphics, the part of the memory reserved for storing the z-axis is called the **"z-buffer"**. It is useful for "hidden line removal" in 2D rendering of a 3D scene: The z-axis is perpendicular to the screen with values increasing towards the viewer. Any point whose z-coordinate is smaller than the corresponding z-buffer value will be hidden behind parts which are already plotted.

DISTANCE. The **distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as

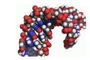
$$\boxed{d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

(Pythagoras). While other distances can be defined like $d(P, Q) = |x - a| + |y - b| + |z - c|$, the Euclidean distance in the above formula is distinguished and natural. Why?

PARITY. We usually work with a **right handed coordinate system**. The photographers coordinate system is an example of a left handed coordinate system. The **"right hand rule"**: thumb=x-direction index finger=y-direction and middle finger=z-direction allows to check that the coordinate system is "right handed".



Parity is relevant in biology (orientation of DNA or Proteins) or particle physics, ("parity violation": physical laws change when we look at them in the mirror). Coordinate systems with different parity can not be rotated into each other. One needs a reflection to do so.

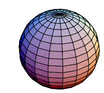


GEOMETRICAL OBJECTS. **curves**, **surfaces** and **bodies** are examples of geometrical objects which can be described using **functions of several variables**. We look at some of them here to get some feel about space. The objects will be treated in more detail later.

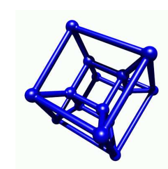
CIRCLE. A **circle** of radius r centered at $P = (a, b)$ is the collection of points which have distance r from P.
SPHERE. A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points which have the distance ρ from P. The equation is $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.
COMPLETION OF SQUARE. The equation $x^2 + bx + c = 0$ is solved by adding $(b/2)^2 - c$ on both sides. This is the **"completion of the square"**.

$$\begin{aligned} x^2 + bx + c &= 0 \\ x^2 + bx + (b/2)^2 &= (b/2)^2 - c \\ (x + b/2)^2 &= (b/2)^2 - c \\ x &= \pm\sqrt{(b/2)^2 - c} - b/2 \,. \end{aligned}$$

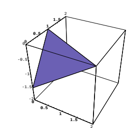
CENTER AND RADIUS OF A SPHERE. The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after completion of the square in each variable equivalent to $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. The equation describes therefore a sphere with **center** $(5/2, 1, 0)$ and **radius** $5/2$.



COORDINATE PLANES, QUADRANTS, OCTANTS. The coordinate axis $x = 0$, $y = 0$ divide the plane into 4 regions called **quadrants**. Similarly, the coordinate planes $x = 0$, $y = 0$ and $z = 0$ divide the space into 8 regions called **octants**. This could be continued into higher dimensions: how many "hyper-regions" are there in four dimensional "hyper-space" which is labeled by points with 4 coordinates (t, x, y, z)? There are 16 hyper-regions and each of them contains one of the 16 points (x, y, z, w), where x, y, z, w are either $+1$ or -1.



DESCRIBING PLANES. To draw the set of all points (x, y, z) which satisfy $x + 2y - 3z = 2$, we first find the intersections with the three coordinate axis. These **intersects** are $P = (2, 0, 0), Q = (0, 1, 0), R = (0, 0, -2/3)$. Then we draw the **traces**, the intersections of the set with the coordinate planes $x = 0$, $y = 0$ or $z = 0$. These three lines bound a triangle in space. Drawing this triangle indicates well the position of the plane.



HISTORICAL. In an appendix to "Geometry" to his "Discours de la méthode" René Descartes (1596-1650) promoted the idea that algebra could be used as a general method to solve geometric problems. In honor of Descartes, rectangular coordinates are also called **Cartesian coordinates**. **Annectote:** "In 1649, Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 a.m. and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace early every morning, Descartes died of pneumonia.