

This is part 3 (of 3) of the homework which is due Tuesday July 3 at the beginning of class.

SUMMARY.

- $\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$. **cross product.**
- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$, where α is the **angle** between vectors. **Area** of parallelogram spanned by \vec{v} and \vec{w} .
- $\vec{v} \times \vec{w}$ is **orthogonal** to \vec{v} and to \vec{w} with length $|\vec{v}||\vec{w}|\sin(\phi)$
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ **triple scalar product**, signed volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.
- $\vec{r}(t) = P + t\vec{v}$ **parametric equation** for a line, P a point, \vec{v} is a vector.
- $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$ **symmetric equation** for a line.
- $\vec{r}(t, s) = P + t\vec{v} + s\vec{w}$ **parametric equation** for a **plane**.
- $ax + by + cz = d$, **implicit equation** for a **plane**.
- Distance Point-Point $d(P, Q) = |\vec{PQ}|$.
- Distance Point-Plane $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$.
- Distance Point-Line $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$.
- Distance Line-Line $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$.

Homework Problems

- 1) (4 points)
- a) (2) Find the cross product \vec{w} of $\vec{u} = (-3, -1, 2)$ and $\vec{v} = (-2, -2, 3)$.
 - b) (1) Find a unit vector \vec{n} orthogonal to \vec{u} and \vec{v} .
 - c) (1) Find the volume of the parallelepiped spanned by \vec{u}, \vec{v} and \vec{w} .

Solution:

- a) $\vec{u} \times \vec{v} = \vec{w} = (1, 5, 4)$
- b) $|\vec{w}| = 42, \vec{n} = \vec{w}/|\vec{w}| = (1/\sqrt{42}, 5/\sqrt{42}, 4/\sqrt{42})$.
- c) $\vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{w} \cdot \vec{w} = |\vec{w}|^2 = 42$.

- 2) (4 points)
- a) (2) Find the distance between the point $(2, -1, 2)$ and the plane $4x - 2y + z = 2$.
 - b) (2) A regular tetrahedron has vertices at the points $P_1 = (0, 0, 3), P_2 = (0, \sqrt{8}, -1), P_3 = (-\sqrt{6}, -\sqrt{2}, -1)$ and $P_4 = (\sqrt{6}, -\sqrt{2}, -1)$. Find the distance between two edges which do not intersect.

Solution:

- a) The point $Q = (0, 0, 2)$ is on the plane. The scalar projection of $P - Q = (2, -1, 0)$ onto the normal vector $(4, -2, 1)$ of the plane is $10/\sqrt{21}$.
- b) The vector $\vec{v} = (P_2 - P_1) = (0, 2\sqrt{2}, -4)$ is parallel to the first edge, the vector $\vec{w} = (P_4 - P_3) = (2\sqrt{6}, 0, 0)$ is parallel to the second edge. The cross product of \vec{v} and \vec{w} is $\vec{n} = (0, -8\sqrt{6}, -8\sqrt{3})$. The distance between the two edges is the scalar projection of $P_3 - P_1$ onto \vec{n} . It is $(P_3 - P_1) \cdot \vec{n}/|\vec{n}| = 2\sqrt{3}$.

- 3) (4 points)
- a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
 - b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.
 - c) Assume you have a triangle in the plane which has edge points with integer coordinates. Show that the area of the triangle is an integer, or half of an integer.

Solution:

- a) Build a triangle with the three vectors u, v, w . Each of the terms in the identity is twice the area of the triangle.
- b) The result is 1.
- c) The formula for the area is $|(A - C) \times (B - C)|/2$.

- 4) (4 points) We are looking for three planes which pass through the origin. We don't know the planes but we see the intersection lines. These lines are spanned by $\vec{u} = \langle 1, 1, 1 \rangle, \vec{v} = \langle 1, 0, 1 \rangle$ and $\vec{w} = \langle 0, 1, 1 \rangle$.
- a) (2 points) Find the equations for the planes.
 - b) (2 points) Verify that in general, if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 1$ and

$$\begin{aligned} \vec{a} &= \vec{v} \times \vec{w} \\ \vec{b} &= \vec{w} \times \vec{u} \\ \vec{c} &= \vec{u} \times \vec{v}, \end{aligned}$$

then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 1$.

Hint. For b), you can use $\vec{b} \times (\vec{u} \times \vec{v}) = (\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$ without proof.

Solution:

- a) We get the normal vectors to the planes by taking the cross product between two different vectors.

$$\begin{aligned} \vec{u} \times \vec{v} &= \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle \\ \vec{u} \times \vec{w} &= \langle 1, 1, 1 \rangle \times \langle 0, 1, 1 \rangle = \langle 0, -1, 1 \rangle \\ \vec{v} \times \vec{w} &= \langle 1, 0, 1 \rangle \times \langle 0, 1, 1 \rangle = \langle -1, -1, 1 \rangle \end{aligned}$$

The equations of the planes are $x - z = 0, -y + z = 0, -x - y + z = 0$.

- b) Focus first on $(\vec{b} \times \vec{c}) = \vec{b} \times (\vec{u} \times \vec{v})$ and use the hint to get $(\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$. The problem asks for the dot product of this with \vec{a} . Now, since \vec{a} is orthogonal to the second term, we obtain $\vec{a}(\vec{b} \cdot \vec{v}) \cdot \vec{u}$. When plugging in the definitions of \vec{a} and \vec{b} , we are left with 1.

- 5) (4 points)
- a) (2) Find the parametric and symmetric equation for the line which passes through the points $P = (1, 2, 3)$ and $Q = (3, 4, 5)$.

b) (2) Find the equation for the plane which contains the three points $P = (1, 2, 3)$, $Q = (3, 4, 4)$ and $R = (1, 1, 2)$.

Solution:

a) The vector $\vec{v} = (2, 2, 2)$ connects the two points. The parametric equation is $P + t\vec{v} = (1, 2, 3) + t(2, 2, 2) = (1 + 2t, 2 + 2t, 3 + 2t)$. The symmetric equation is $(x - 1)/2 = (y - 2)/2 = (z - 3)/2$.

b) A normal vector $\vec{n} = (1, -2, 2) = (a, b, c)$ of the plane $ax + by + cz = d$ is obtained as the cross product of $P - Q$ and $R - Q$. With $d = \vec{n} \cdot P = 3$, we have the equation $x - 2y + 2z = 3$.

b0

Remarks

(You don't need to read these remarks to do the problems.)

To 4): three vectors with nonzero triple scalar product are called **non-coplanar**. Adding integer multiples of such vectors form a **lattice** $n\vec{u} + m\vec{v} + k\vec{w}$, where n, m, k are integers.

The vectors \vec{a}, \vec{b} and \vec{c} in the problem define a new lattice which is called the **reciprocal lattice**. Crystallographers also denote them by \vec{a}^*, \vec{b}^* and \vec{c}^* . The reciprocal lattice is essential for the study of crystal lattices and their diffraction properties obtained by shooting X-rays onto them.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Prove the following identity for vectors $\vec{a}, \vec{b}, \vec{c}$ in space:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

2) Find a general formula for the volume of a tetrahedron with corners P, Q, R, S .

Hint. Find first a formula for the area of one of its triangular faces, and then a formula for the distance from the fourth point to that face.

The change of the angular momentum satisfies the formula

$$\frac{d}{dt}\vec{L} = \vec{L} \times \vec{\Omega},$$

3) where $\vec{\Omega}$ is the angular velocity vector. Verify that the length of \vec{L} does not change in time. To do so, compute the time derivative $\vec{L} \cdot \vec{L}$ and show that it is 0. You can use

$$\frac{d}{dt}[\vec{v}(t) \times \vec{w}(t)] = \vec{v}'(t) \times \vec{w}(t) + \vec{v}(t) \times \vec{w}'(t).$$

