

7/10/2007 CURVES

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This is part 1 (of 2) of the homework for the third week. It is due July 17 at the beginning of class.

SUMMARY.

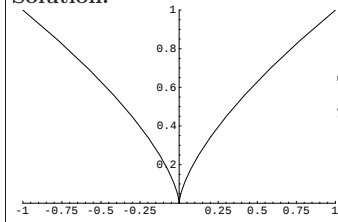
- $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a, b]$ **curve** in space.
- $t \mapsto \vec{r}(t)$ is called **parametric representation** of the curve.
- $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ **velocity**, $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$ **unit tangent vector**.
- $\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$ **acceleration**.
- $\vec{R}(t) = \int_a^t \vec{r}(s) ds = (\int_a^t x(s) ds, \int_a^t y(s) ds, \int_a^t z(s) ds) + (C_1, C_2, C_3)$ **anti-derivative**, satisfies $\vec{R}'(t) = \vec{r}(t)$. (C_1, C_2, C_3) are arbitrary constants.
- $s(t)$ function, $\vec{r}(t)$ curve, $\frac{d}{dt}r(s(t)) = r'(s(t))s'(t)$. **scalar chain rule**.

Homework Problems

1) (4 points)

Sketch the plane curve $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle t^3, t^2 \rangle$ for $t \in [-1, 1]$ by plotting the points for different values of t . Calculate its velocity $\vec{r}'(t)$ as well as its acceleration $\vec{r}''(t)$ at the point $t = 2$.

Solution:



The velocity is $\vec{r}'(t) = \langle 3t^2, 2t \rangle$. The acceleration is $\vec{r}''(t) = \langle 6t, 2 \rangle$. At the time $t = 2$ we have $\vec{r}'(2) = \langle 12, 4 \rangle$ and $\vec{r}''(2) = \langle 12, 2 \rangle$.

2) (4 points) A device in a car measures the acceleration $\vec{r}''(t) = \langle \cos(t), -\cos(3t) \rangle$ at time t . Assume that the car is at the origin at time $t = 0$ and has zero speed at $t = 0$, what is its position $\vec{r}(t)$ at time t ?

Solution:

$\vec{r}'(t) = \langle \sin(t), -\sin(3t)/3 \rangle + (C_1, C_2)$. Because the car has zero speed at time $t = 0$, we have $C_1 = C_2 = 0$.

From $\vec{r}'(t) = \langle \sin(t), -\sin(3t)/3 \rangle$, we obtain $\vec{r}(t) = \langle -\cos(t), \cos(3t)/9 \rangle + (C_1, C_2)$. Because $\vec{r}(0) = (0, 0)$, we have $C_1 = 1, C_2 = -1/9$. $\vec{r}(t) = \langle -\cos(t) + 1, \cos(3t)/9 - 1/9 \rangle$.

3) (4 points) Consider the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle t^2, 1 + t, 1 + t^3 \rangle$.

a) (1) Verify that it passes through the point $(1, 0, 0)$.

b) (3) Find the velocity vector $\vec{r}'(t)$, the acceleration vector $\vec{r}''(t)$ as well as the jerk vector $\vec{r}'''(t)$ at the point $(1, 0, 0)$.

Solution:

a) Take $t = -1$.

b) $\vec{r}'(t) = \langle 2t, 1, 3t^2 \rangle$,

$\vec{r}''(t) = \langle 2, 0, 6t \rangle$,

$\vec{r}'''(t) = \langle 0, 0, 6 \rangle$

At time $t = -1$, we obtain $\vec{r}'(-1) = \langle -2, 1, 3 \rangle$,

$\vec{r}''(-1) = \langle 2, 0, -6 \rangle$,

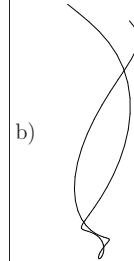
$\vec{r}'''(-1) = \langle 0, 0, 6 \rangle$

4) (4 points) a) (2) Verify that the curve $\vec{r}(t) = \langle t \cos(t), 2t \sin(t), t^2 \rangle$ is located on the elliptic paraboloid $z = x^2 + y^2/4$.

b) (2) Use this fact to sketch the curve.

Solution:

a) Just plug in $x(t)^2 + y(t)^2 = z$.



5) (4 points) Find the parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ of the curve obtained by intersecting the elliptical $x^2/9 + y^2/4 = 1$ with the surface $z = xy$.

a) Write down the formula for the velocity vector $\vec{r}'(t)$.

b) If $s(t) = t^2$. Find $\frac{d}{dt}r(s(t))$ using the chain rule.

Solution:

We find first $x(t) = 3 \cos(t), y(t) = 2 \sin(t)$ using the first equation. Then get $z(t) = x(t)y(t) = 6 \cos(t) \sin(t)$.

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 3 \cos(t), 2 \sin(t), 6 \cos(t) \sin(t) \rangle$. The velocity vector is $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle -3 \sin(t), 2 \cos(t), 6 \cos^2(t) - 6 \sin^2(t) \rangle$.

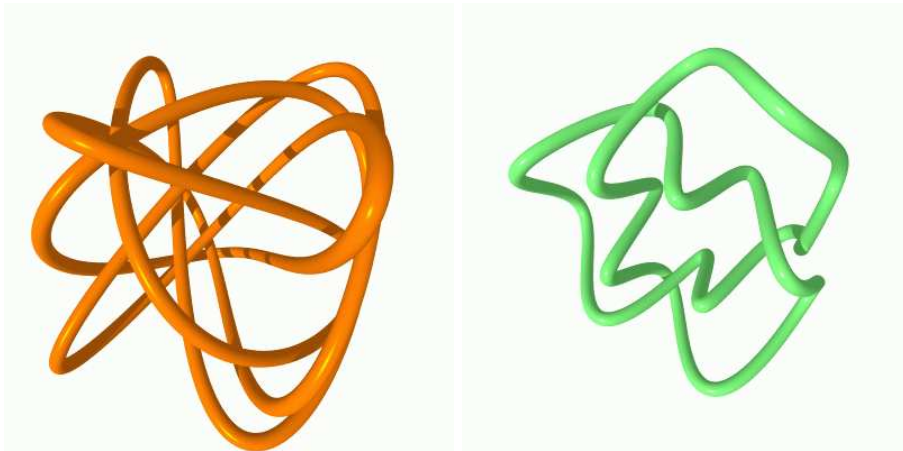
b) $d/dtr(s(t)) = r'(s^2)2t = \langle -3 \sin(t), 2 \cos(t), 6 \cos^2(t) - 6 \sin^2(t) \rangle 2t$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) A closed curve in space is called a **knot**. Consider the space curve $\vec{r}(t) = (\sin(3t), \cos(4t), \cos(5t))$. Find the smallest interval $[a, b]$ such that this curve is a knot. Sketch the curve.

Hint. Try first without technology. If needed, peek at the website <http://www.math.harvard.edu/jdg> or type `ParametricPlot3D[{Sin[3t], Cos[4t], Cos[5t]}, {t, 0, 2Pi}]` in Mathematica.



- 2) How could one verify that it is not possible to deform the knot $\vec{r}(t) = (\sin(3t), \cos(4t), \cos(5t))$ into the trivial knot $\vec{r}(t) = (\cos(t), \sin(t), 0)$ in such a way that during the deformation, the curve can never self-intersect?

Hint. Look at the possible types of closed curves which don't intersect the knot. How many different types are there for the trivial knot or for the given knot?