7/11/2007 ARC LENGTH/CURVATURE/GRADIENT **O.Knill,Maths21a**

This is part 2 (of 2) of the homework for the third week. It is due July 17 at the beginning of class.

SUMMARY.

- **speed** $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.
- **Unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$.
- **Unit normal vector** $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$
- **Binormal vector** $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.
- $\int_a^b |\vec{r}'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$ **arc length**.
- $|\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$ **curvature**.
- Example of graph: $\vec{r}(t) = (t, f(t))$, $\kappa(t) = f''(t)/(1 + f'(t)^2)^{3/2}$.
- f_x, f_y, f_z partial derivatives.
- $\nabla f(x, y) = (f_x(x, y), f_y(x, y))$ **gradient**.
- $\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z))$ **gradient**.

Homework Problems

1) (4 points) Find the arc length of the curve $\vec{r}(t) = (t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t))$, $0 \leq t \leq \pi$.

Solution:
The velocity is $\vec{r}\,'(t) = (2t, t\sin(t), t\cos(t))$ and the speed is $|\vec{r}\,'(t)|\sqrt{5t^2} = \sqrt{5}t$. The arc length of the curve is $\int_0^\pi \sqrt{5}t\,dt = \pi^2\sqrt{5}/2$.

2) (4 points) Find the curvature of $\vec{r}(t) = (e^t\cos(t), e^t\sin(t), t)$ at the point $(1, 0, 0)$.

Solution:
To use the formula $\kappa(t) = |\vec{r}\;'(t) \times \vec{r}\;''(t)|/|\vec{r}\;'(t)|^3$, we need to know the velocity $\vec{r}\;'(t) = (e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t)), 1)$, as well as the acceleration $\vec{r}\;''(t) = (-2e^t\sin(t), 2e^t\cos(t), 0)$. At the time $t = 0$, we have $\vec{r}\;'(0) = (1, 1, 1)$ and $\vec{r}\;''(t) = (0, 2, 0)$. Now apply the formula $\kappa(0) = |(1, 1, 1) \times (0, 2, 0)|/\sqrt{3}^3 = |(-2, 0, 2)|/\sqrt{3}^3 = \sqrt{8}/\sqrt{3}^3 = 2\sqrt{6}/9$.

3) (4 points)
(3) Find the vectors $\vec{T}(t)$, $\vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = (t^2, t^3, 0)$ for $t = 2$.
(1) Do the vectors depend continuously on t for all t?

Solution:
a) $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)| = (1, 2, 0)/\sqrt{10}$, $\vec{N}(t) = \vec{T}\,'(t)/|\vec{N}\,'(t)| = (-3, 1, 0)/\sqrt{10}$, $\vec{B}(t) = T(t) \times N(t) = (0, 0, 1)$.
b) The $\vec{T}$ and $\vec{N}$ vectors do depend continuously on t. While $T'(t) = (2t, 3t^2, 0)/\sqrt{4t^2 + 9t^4}$ looks discontinues at $t = 0$ at first, one can divide that formula by t to get $T(t) = (2, 3t, 0)/\sqrt{4 + 9t^2}$ which is smooth in t. Also the second derivative $T'(t)$ is smooth. The third vector, as a cross product depends continuously on t also.

4) a) (2 points) Let $f(x, y, z) = x^2 + 3y^4 + \sin(z)$.
Find all the partial derivatives f_x, f_y, f_z.
b) (1 point) Write down the gradient $\nabla f(x, y, z)$.
c) (1 point) Let $\vec{r}(t) = (t^2, \sqrt{t}, t)$. Verify the **chain rule formula** $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{d}{dt} f(\vec{r}(t))$ in this case.

Solution:
a) $f_x = 2x$, $f_y = 12y^3$, $f_z = \cos(z)$
b) $\nabla f(x, y, z) = (2x, 12y^3, \cos(z))$.
c) $r'(t) = (2t, 1/(2\sqrt{t}), 1)$ and $f(r(t)) = 3t^2 + t^4 + \sin(t)$. $\nabla f(r(t)) = (2x, 12y^3, \cos(z))$ and $\nabla f(r(t)) \cdot r'(t) = 6t + 4t^3 + \cos(t)$ The direct computation of the derivative of $f(r(t))$ gives the same result.

5) (4 points) Verify that $f(x, t) = e^{-rt}\sin(x + ct)$ satisfies the PDE $f_t(x, t) = cf_x(x, t) - rf(x, t)$ called the **advection equation**.

Solution:
Differentiate $f_x(x, y) = e^{-rt}\cos(x + ct)$ and $f_t(x, y) = -re^{-rt}\sin(x + ct) + ce^{-rt}\cos(x + ct)$ and compare.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Let $\vec{r}(t) = (t, t^2)$. Find the equation for the **caustic** $\vec{s}(t) = \vec{r}(t) + \vec{N}(t)/\kappa(t)$ known also as the **evolute** of the curve.

2) Find the evolute of the curve $\vec{r}(t) = (t, t^4)$.

3) If $\vec{r}(t) = (-\sin(t), \cos(t))$ is the boundary of a coffee cup and light enters in the direction $(-1, 0)$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. Find a parameterization of this curve.