

7/24/2007 DOUBLE INTEGRALS O.Knill, Maths21a This is part 1 (of 2) of the weekly homework. It is due July 31 at the beginning of class.

SUMMARY.  $dA = dxdy$  is called **area element**.

- $\int \int_R f dA = \int_a^b \int_c^d f(x, y) dydx$  is called a **double integral** over a rectangle  $R$ .
- $\int \int_R f dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dydx$  double integral over a **type I region**.
- $\int \int_R f dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) dxdy$  double integral over a **type II region**.
- $A(R) = \int \int_R 1 dA$  is called the **area** of  $R$ .
- $\frac{1}{A(R)} \int \int_R f dA$  is called the **average value** or the **mean** of  $f$  on  $R$ .
- For  $f \geq 0$ , the integral  $\int \int_R f dA$  is the volume of the solid over  $R$  bounded below by the  $xy$ -plane and bounded above by the graph of  $f$ .

## Homework Problems

- 1) (4 points) Calculate the iterated integral  $\int_1^4 \int_0^2 (2x - \sqrt{y}) dxdy$ . Can you interpret it as a volume of a solid? If not, can you express the result in terms of two volumes?

**Solution:**

Start with the inner integral  $\int_0^2 (2x - \sqrt{y}) dx = 4 - 2\sqrt{y}$ . Integrating this from 1 to 4 gives  $\frac{8}{3}$ . The graph of  $2x = \sqrt{y}$  passes through the rectangle so that the function is positive somewhere in the rectangle and negative elsewhere. The integral is not a volume but the difference between two positive volumes.

- 2) (4 points) Find the area of the region

$$R = \{(x, y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value of  $f(x, y) = y$  over that region.

**Remark.** You will use here the integral  $\int_0^{2\pi} \sin^2(x) dx$  treated in class.

**Solution:**

In this problem, it helps to see that  $\int_0^{2\pi} \cos(x) dx = 0$  and  $\int_0^{2\pi} \cos^2(x) dx = \pi$  and the same for  $\sin$ .

The area is  $A = \int_0^{2\pi} \int_{\sin(x)-1}^{\cos(x)+2} 1 dxdy = \int_0^{2\pi} (\cos(x)+2) - (\sin(x)-1) dx = 6\pi$ . The average value is  $\int_0^{2\pi} \int_{\sin(x)-1}^{\cos(x)+2} y dxdy / A = \int_0^{2\pi} (\cos(x)+2)^2 - (\sin(x)-1)^2 dx / A = (4-1)\pi / (2\pi) = 3\pi / (6\pi) = \frac{1}{2}$ .

- 3) (4 points) Find the volume of the solid lying under the paraboloid  $z = x^2 + y^2$  and above the rectangle  $R = [-2, 2] \times [-3, 3] = \{(x, y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\}$ .

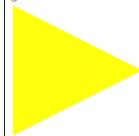
**Solution:**

We have to compute the double integral of  $f(x, y) = x^2 + y^2$  over  $R$ . The inner integral is  $\int_{-3}^3 (x^2 + y^2) dy = 18 + 6y^2$  so that  $\int_{-2}^2 \int_{-3}^3 (x^2 + y^2) dydx = \int_{-2}^2 (18 + 6y^2) dy = 104$ .

- 4) (4 points) Calculate the iterated integral  $\int_0^1 \int_x^{2-x} (x^2 - y) dydx$ . Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

**Solution:**

$\int_0^1 \int_x^{2-x} (x^2 - y) dydx = -5/6$ . The region is a triangle bound by the lines  $y = x$ , the line  $y = 2 - x$  and the  $y$  axis. The inner integral is  $-2 + 2x + 2x^2 - 2x^3$ .



As a type II region, the region has to be split  $\int_0^1 \int_0^y (x^2 - y) dxdy + \int_1^2 \int_0^{2-x} (x^2 - y) dxdy = -1/4 - 7/12 = -5/6$ .

- 5) (4 points) Compute the probability that a quantum particle with energy  $(k^2 + n^2)\hbar^2 / (2m) = 5\hbar^2 / (2m)$  is in the region  $R = [0, \pi/2] \times [0, \pi/2] = \{(x, y) \mid 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\}$  of the square box  $[0, \pi] \times [0, \pi]$ .

Check. The result will involve integrals

$$A = \int_0^\pi \int_0^\pi \sin^2(kx) \sin^2(ny) dxdy$$

as well as

$$B = \int_0^{\pi/2} \int_0^{\pi/2} \sin^2(kx) \sin^2(ny) dxdy$$

In the notes, you find an analogues computation for a different region  $R$  and a different energy level. Identities like  $1 - 2\sin^2(x) = \cos(2x)$  are useful here.

**Solution:**

The integral  $A = \pi^2 / 4$  is the integral of the wave function over the whole box, the integral  $B = \pi^2 / 16$  is the integral over the region  $R$ . The result is  $B/A = 1/4$ . This result makes sense due to the symmetry of the wave.

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## Remarks

(You don't need to read these remarks to do the problems.)

A quantum mechanical particle confined to a region  $X$  in two dimensions is represented by a function  $f(t, x, y)$  satisfying  $\int \int_X f^2(t, x, y) \, dx dy = 1$ . For each time  $t$ , the probability that the particle is in some subregion  $R$  is the double integral

$$\int \int_R f^2(t, x, y) \, dx dy .$$

According to the classical interpretation of quantum mechanics, particles don't have determined positions any more, the probabilities are all an experimenter can measure. If a particle is exposed to an external field  $F(x, y) = \nabla V(x, y)$ , then the evolution of the particle is given by the equation

$$\frac{i\hbar}{m} f_t(t, x, y) = f_{xx}(x, y) + f_{yy}(x, y) + V(x, y)f(x, y)$$

which is called the **Schrödinger equation**. A special case is if there is no force  $F = \nabla V$ . In that case, one can assume that  $V(x, y) = 0$ . Important are solutions  $f(x, y)$  which satisfy the partial differential equation

$$f_{xx}(x, y) + f_{yy}(x, y) + V(x, y)f(x, y) = Ef(x, y) ,$$

where  $E$  is a number called the **energy**. In this case, the evolution of the particle is  $\frac{i\hbar}{m} f_t(t, x, y) = Ef(t, x, y)$  which has a solution  $e^{i\hbar Et/m} f(0, x, y)$ . It is a mathematical fact that for a bounded region  $X$ , not all energies  $E$  are allowed. They come in discrete steps, energies appear "**quantized**". Quantized energies also appear for potentials like  $V(x, y) = x^2 + y^2$ , which is called the quantum mechanical oscillator or  $V(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$  which is called the Coulomb potential. All what has been said works also in three dimensions. It is just that  $f$  depends now on three space variables and the double integrals will be replaced by triple integrals. If  $V(x, y, z) = 1/|(x, y, z)|$ , then the mathematics of the solutions to  $f_{xx}(x, y, z) + f_{yy}(x, y, z) + f_{zz}(x, y, z) + V(x, y, z)f(x, y, z) = Ef(x, y, z)$  is the story of the **hydrogen atom**. The possible energy levels explains to a great deal the build-up of the periodic elements and so the stuff we are made of.

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## Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Let  $M$  be a polygon in the plane where each edge is at a lattice point. Verify that the area  $A$  of the polygon satisfies  $A = I + B/2 - 1$ , where  $I$  is the number of lattice points inside the polygon and  $B$  is the number of lattice points at the boundary.

**Solution:**

- 2) The integral  $\int_0^1 \arccos(\sqrt{x}) \, dx$  can be written as a double integral  $\int_0^1 \int_0^{\arccos(\sqrt{x})} dy dx$ . Calculate this integral.