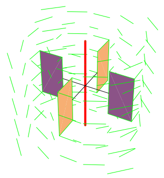
CURL AND DIV	Maths21a, O. Knill

CURL (3D). The curl of a 3D vector field $F = (P,Q,R)$ is defined as the 3D vector field

$$\mathrm{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) \; .$$

CURL (2D). Recall, the curl of a 2D vector field $F = (P,Q)$ is $Q_x - P_y$, a scalar field.



WANTED! Is there a multivariable calculus book, in which the above wheel is **not** shown? The wheel indicates the curl vector if F is thought of as a wind velocity field. As we will see later the direction in which the wheel turns fastest, is the direction of curl(F). The wheel could actually be used to measure the curl of the vector field at each point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl.

DIV (3D). The **divergence** of $F = (P,Q,R)$ is the scalar field $\mathrm{div}(P,Q,R) = \nabla \cdot F = P_x + Q_y + R_z$.
DIV (2D). The **divergence** of a vector field $F = (P,Q)$ is $\mathrm{div}(P,Q) = \nabla \cdot F = P_x + Q_y$.

NABLA CALCULUS. With the "vector" $\nabla = (\partial_x, \partial_y, \partial_z)$, we can write $\mathrm{curl}(F) = \nabla \times F$ and $\mathrm{div}(F) = \nabla \cdot F$. This is both true in 2D and 3D.

LAPLACE OPERATOR. $\Delta f = \mathrm{div}\mathrm{grad}(f) = f_{xx} + f_{yy} + f_{zz}$ can be written as $\nabla^2 f$ because $\nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f))$. One can extend this to vectorfields $\Delta F = (\Delta P, \Delta Q, \Delta R)$ and writes $\nabla^2 F$.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with Nabla calculus from formulas for vectors like $\vec{v} \times \vec{v} = \vec{0}, \vec{v} \cdot \vec{v} \times \vec{w} = 0$ or $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$.

$\mathrm{div}(\mathrm{curl}(F)) = 0$.
$\mathrm{curlgrad}(F) = \vec{0}$
$\mathrm{curl}(\mathrm{curl}(F)) = \mathrm{grad}(\mathrm{div}(F) - \Delta(F))$.

$\nabla \cdot \nabla \times F = 0$.
$\nabla \times \nabla F = \vec{0}$.
$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F$.

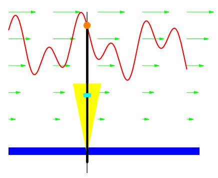
QUIZZ. Is there a vector field G such that $F = (x + y, z, y^2) = \mathrm{curl}(G)$? Answer: no because $\mathrm{div}(F) = 1$ is incompatible with $\mathrm{div}I\mathrm{curl}(G)) = 0$.

ADDENDA TO GREEN'S THEOREM. Green's theorem, one of the advanced topics in this course is useful in physics. We have already seen the following applications

- Simplify computation of double integrals
- Simplify computation of line integrals
- Formulas for centroid of region
- Formulas for area
- Justifying, why mechanical integrators like the planimeter works.

APROPOS PLANIMETER.

The **cone planimeter** is a mechanical instrument to find the antiderivative of a function $f(x)$. It uses the fact that the vector field $F(x,y) = (y,0)$ has $\mathrm{curl}(F) = -1$. By Greens theorem, the line integral around the type I region bounded by 0 and the graph of $f(x)$ in the counter clockwise direction is $\int_a^x f(x)\; dx$. The planimeter determines that line integral.



THERMODYNAMICS. Gases or liquids are often described in a $P - V$ **diagram**, where the volume in the x-axis and the pressure in the y axis. A periodic process like the pump in a refrigerator leads to closed curve $\gamma : r(t) = (V(t), p(t))$ in the $V - p$ plane. The curve is parameterized by the time t. At a given time, the gas has volume $V(t)$ and a pressure $p(t)$. Consider the vector field $F(V,p) = (p, 0)$ and a closed curve γ. What is $\int_\gamma F\; ds$? Writing it out, we get $\int_0^t (p(t), 0) \cdot (V'(t), p'(t)\; dt = \int_0^t p(t)V'(t)dt = \int_0^t pdV$. The curl of $F(V,p)$ is -1. You see by Green's theorem the integral $-\int_0^t pdV$ is the area of the region enclosed by the curve.

MAXWELL EQUATIONS (in homework you assume no current $j = 0$ and charges $\rho = 0$. c = speed of light.

$\mathrm{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\mathrm{curl}(E) = -\frac{1}{c} B_t$	Faraday's law	change of magnetic flux induces voltage
$\mathrm{curl}(B) = \frac{1}{c} E_t + \frac{4\pi}{c} j$	Ampère's law	current or change of E produces magnetic field
$\mathrm{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

2D MAXWELL EQUATIONS? In space dimensions d different than 3 the electromagnetic field has $d(d+1)/2$ components. In 2D, the magnetic field is a scalar field and the electric field $E = (P,Q)$ a vector field. The 2D Maxwell equations are $\mathrm{curl}(E) = -\frac{1}{c}\frac{d}{dt} B, \mathrm{div}(E) = 4\pi\rho$. Consider a region R bounded by a wire γ. Green's theorem tells us that $d/dt \int\int_R B(t)\; dxdy$ is the line integral of E around the boundary. But $\int_\gamma Eds$ is a voltage. A change of the magnetic field produces a voltage. This is the **dynamo** in 2D. We will see the real dynamo next week in 3D, where electromagnetism works (it would be difficult to generate a magnetic field in flatland). If v is the velocity distribution of a fluid in the plane, then $\omega(x,y) = \mathrm{curl}(v)(x,y)$ is the **vorticity** of the fluid. The integral $\int\int_R \omega dxdy$ is called the **vortex flux** through R. Green's theorem assures that this flux is related to the circulation on the boundary.

FLUID DYNAMICS. v velocity, ρ density of fluid.

Continuity equation	$\dot{\rho} + \mathrm{div}(\rho v) = 0$	no fluid get lost
Incompressibility	$\mathrm{div}(v) = 0$	incompressible fluids, in 2D: $v = \mathrm{grad}(u)$
Irrotational	$\mathrm{curl}(v) = 0$	irrotation fluids

A RELATED THEOREM. If we rotate the vector field $F = (P,Q)$ by 90 degrees $= \pi/2$ we get a new vector field $G = (-Q, P)$. The integral $\int_\gamma F \;\cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with the length of dr. With $\mathrm{div}(F) = (P_x + Q_y)$, we see that $\mathrm{curl}(F) = \mathrm{div}(G)$. Green's theorem now becomes

$$\int\int_R \mathrm{div}(G)\; dxdy = \int_\gamma G \cdot dn \; ,$$

where $dn(x,y)$ is a normal vector at (x,y) orthogonal to the velocity vector $r'(x,y)$ at (x,y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. **It is in two dimensions just Green's theorem disguised.** There are only 2 basic integral theorems in the plane: Green's theorem and the FTLI.

PREVIEW. Green's theorem is of the form $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are 2 such theorems in dimensions 2, three theorems in dimensions 3, four in dimension 4 etc. In the plane, Green's theorem is the second one besides the fundamental theorem of line integrals FTLI. In three dimensions, there are two more theorems beside the FTLI: Stokes and Gauss Theorems which we will see in the next week.

dim	dim(R)	theorem
1D	1	Fund. thm of calculus

2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem
3D	3	Gauss theorem

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence