

## 8/7/2007 CURL AND GREENS THEOREM

O.Knill, Maths21a

This is part 1 (of 3) of the weekly homework. It is due Friday August 10 at the global review in lecture hall A at 7 PM.

### SUMMARY.

- $F(x, y) = (P(x, y), Q(x, y))$  **vector field in the plane.**
- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$  **line integral** of  $F$  along curve  $C : t \mapsto r(t)$ .
- Example:  $C : r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$  (circle),  $F(x, y) = (-y, x)$ .  $\int_C F \cdot dr = \int_F(r(t)) \cdot r'(t) dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} 1 dt = 2\pi$ .
- $\text{curl}(P, Q) = Q_x - P_y$ . **Curl** for 2D vector fields.
- $C$  **positively oriented boundary** of the region  $D$ :  
(the region is "to the left" when you follow the boundary).
- $\int_C F \cdot dr = \int_D \text{curl}(F) dA$  **Greens theorem.**  
Written out:  $\int_C F(r(t)) \cdot r'(t) dt = \int_D (Q_x - P_y) dx dy$ .
- $\int_C x dy$  **area** of  $D$ ,  $C$  is the positively oriented boundary of  $D$ :  
Example:  $C$  unit circle:  $x(t) = \cos(t), dy = \cos(t) dt$ , Area =  $\int_0^{2\pi} \cos^2(t) dt = \pi$ .

## Homework Problems

- 1) (4 points) Calculate the line integral  $\int_C 2(y + x \sin(y), x^2 \cos(y) - 3y^2) dr$  along a triangle  $C$  with edges  $(0, 0)$ ,  $(1, 0)$  and  $(1, 1)$  using Green's theorem.

### Solution:

$\text{curl}(F)(x, y) = 4x \cos(y) - 2 - 2x \cos(y) = 2x \cos(y) - 2$ . By Green's theorem, we have to integrate this function over the region  $R$  enclosed by the triangle:

$$\int_0^1 \int_0^x 2x \cos(y) - 2 dy dx = \int_0^1 (2x \sin(x) - 2x) dx = -1 - 2 \cos(1) + 2 \sin(1).$$

- 2) (4 points) Evaluate the line integral of the vector field  $F(x, y) = (xy^2, x^2)$  along the rectangle with vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(2, 3)$ ,  $(0, 3)$  in two ways. Do this by calculating the line integral as well as using Greens theorem.

### Solution:

Integrating  $\text{curl}(F) = 2x - 2xy$  over the rectangle gives

$$\int_0^2 \int_0^3 2x - 2xy dy dx = \int_0^2 6x - 9x dx = -12/2 = -6$$

The line integral is a sum of four line integrals:  $\int_0^2 (t0^2, t^2) \cdot (1, 0) dt = 0$  and  $\int_0^3 (2t^2, 4) \cdot (0, 1) dt = 12$  and  $-\int_0^2 (9t, t^2) \cdot (1, 0) dt = -18$  as well as  $\int_0^3 (0t^2, 0) \cdot (0, 1) dt = 0$ . The sum is also  $-6$ .

- 3) (4 points) Find the area of the region bounded by the hypocycloid  $\vec{r}(t) = (\cos^3(t), \sin^3(t))$  using Green's theorem. The curve is parameterized by  $t \in [0, 2\pi]$ .

### Solution:

Take a vector field  $F(x, y) = (0, x)$  which has the curl 1. Then by Green the area is the line integral  $\int_0^{2\pi} (0, \cos^3(t)) \cdot (-3 \cos^2(t) \sin(t), 3 \sin^2(t) \cos(t)) dt = 3 \int_0^{2\pi} \cos^4(t) \sin^2(t) dt = 3 \int_0^{2\pi} \sin^2(2t)/4(\cos(2t) + 1)/2 = 3/8\pi$ .

- 4) (4 points) Let  $F(x, y) = (-y/(x^2 + y^2), x/(x^2 + y^2))$ . Let  $C : \vec{r}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ .
- What is  $\int_C F \cdot dr$ ?
  - Show that  $\text{curl}(F) = 0$  everywhere for  $(x, y) \neq (0, 0)$ .
  - Let  $f(x, y) = \arctan(y/x)$ . Verify that  $\nabla f = F$ .
  - Why do a) and b) not contradict the fact that a gradient field has the closed loop property?

Hint: is  $f$  really a continuous potential?

### Solution:

- $2\pi$ .
- Direct differentiation  $Q_x(x, y) - P_y(x, y) = 0$ .
- Use  $\arctan'(x) = 1/(1 + x^2)$ .
- The function  $f(x, y)$  is not continuous everywhere. Also the vector field  $F(x, y)$  is not smooth everywhere. There is a singularity at  $(0, 0)$ .

- 5) a) Verify that if  $C$  is the line segment connecting the point  $(x_1, y_1)$  to the point  $(x_2, y_2)$ , and  $F$  is the vector field  $F(x, y) = (-y, x)$  then  $\int_C F \cdot dr = (x_1 y_2 - x_2 y_1)$ .

- b) Use a) to verify that if  $(x_1, y_1), \dots, (x_n, y_n)$  are the vertices of a polygon in the plane, then  $A = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \dots + (x_n y_1 - x_1 y_n)]$  is the area of the polygon.

### Solution:

Parametrize the segment by  $\vec{r}(t) = (x_1, y_1) + t(x_2 - x_1, y_2 - y_1)$  and get  $F(\vec{r}(t)) = (-y_1 - t(y_2 - y_1), x_1 + t(x_2 - x_1))$  and  $\vec{r}'(t) = (x_2 - x_1, y_2 - y_1)$ . So,  $\int_0^1 F(r(t)) \cdot r'(t) dt = (y_2 - y_1)(x_1 + (x_2 - x_1)^2/2 + (x_2 - x_1)(y_1 + (y_2 - y_1)^2/2)$ .

- b) Sum the result in a) from  $i = 1$  to  $i = n$ .

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## Remarks

(You don't need to read these remarks to do the problems.)

To problem 4: This vector field is important in fluid dynamics. It models a single **vortex**. The curl of  $F$  is zero everywhere except at the origin. Physicists would say that that  $\text{curl}(F)$  is a "delta function" located at the origin. In fluid dynamics, one can model fluids using a finite set of vortices. This homework problem is also crucial if you want to solve Nash's problem.

To problem 5: This formula is actually used in applications like for example in ray tracing which is a CPU time intensive task. The software has to compute the light ray paths bouncing around in a virtual world, compute reflections or refractions. Tracing an image can take from a few seconds to days. A single frame in movies like "Toy Story" took several hours to render. To get the large number of frames needed for a movie companies like "Pixar" (recently again visible with "Finding Nemo") use "computer farms" a huge number of workstations.

To compute the normal vector to a polygon, an area formula is used which is derived from Green's theorem. This formula is also used to compute normals to a surface. Computing the normal from three points on the surface only is error prone. It is often better to consider a polygon  $P_i = (x_i, y_i, z_i)$  on the surface. What is the normal to such a polygon? Note that the points  $P_i$  are not necessarily on a plane. The  $xy$  projection of the polygon gives a polygon  $(x_i, y_i)$  in the plane which has the area  $1/2 \sum_k (x_{k+1} + x_k)(y_{k+1} - y_k)$  a formula derived from Green's theorem.

The normal vector to a not necessarily planar polygon  $P_i = (x_i, y_i, z_i)$  in space is defined as

$$n = \begin{bmatrix} 1/2 \sum_k (y_{k+1} + y_k)(z_{k+1} - z_k) \\ 1/2 \sum_k (z_{k+1} + z_k)(x_{k+1} - x_k) \\ 1/2 \sum_k (x_{k+1} + x_k)(y_{k+1} - y_k) \end{bmatrix}.$$



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## Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) The planimeter calculates the area: the **planimeter vector field**  $F(x, y) = (P(x, y), Q(x, y))$  is defined by attaching a unit vector orthogonal to the vector  $(x-a, y-b)$  at  $(x, y)$ , where  $(a, b)$  is the "knee" of the planimeter. The wheel rotation is the line integral of  $F$  along the boundary of  $R$ . By **Green's theorem**, this integral is the double integral of  $\text{curl}(F)$  over  $R$ . The planimeter vector field is explicitly given by  $F(x, y) = (P(x, y), Q(x, y)) = (-(y - b(x, y)), (x - a(x, y)))$ . Furthermore,  $\text{curl}(F) = Q_x - P_y$  is equal to  $2 + (-a_x - b_y)$  which is 2 plus the curl of the vector field  $G(x, y) = (b(x, y), -a(x, y))$ . Show that  $\text{curl}(G) = -1$ . For more information see <http://www.math.duke.edu/education/ccp/materials/mvcalc/green/>
- 2) Let  $D$  be a region bounded by a simple closed path  $C$  in the plane. Use Green's theorem to prove that the coordinates of the center of mass are  $(\int_C x^2 dy / (2A), -\int_C y^2 dx / (2A))$ , where  $A$  is the area of  $D$ .