



You do not need a book. If you want a second opinion, I recommend

**James Stewart**, Multivariable Calculus, 5th edition 2009, ISBN-10:0-495-56054-5

ORGANISATION

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Lectures and Section Hall E



IMPORTANT DATES

1. EXAM	2. EXAM	FINAL
JULY 8	JULY 22	AUG 5
8:30 AM	8:30 AM	8:30 AM
SC HALL E	SC HALL E	SC HALL E

GRADES

PART	GRADE 1
1. HOURLY	20
2. HOURLY	20
HOMEWORK	25
LAB	5
FINAL	30

# MATHS 21A

## SYLLABUS 2010

This standard multivariable calculus course extends single variable calculus to higher dimensions. It provides a vocabulary for understanding fundamental equations of nature like weather, planetary motion, waves, heat, finance, or quantum mechanics. It teaches important background needed for statistics, computer graphics, bioinformatics, etc. It provides tools for describing curves, surfaces, solids and other geometrical objects in three dimensions. It develops methods for solving optimization problems with and without constraints. You learn a powerful computer algebra system. The course will enhance problem solving skills and prepares you for further study in other fields of mathematics and its applications.

# CALENDAR

# SYLLABUS

# CHEKLIST

SU	MO	TU	WE	TH	FR	SA
20	21	22	23	24	25	26
27	28	29	30	1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	27	29	30	31
1	2	3	4	5	6	7

## 1. Week: Geometry and Space

Lect 1-2 6/22 Space, Vectors, Dot Product

Lect 3-4 6/24 Cross product, Lines/Planes, Distances

## 2. Week: Surfaces and Curves

Lect 5-6 6/29 Implicit and Parametric Surface

Lect 7-8 7/1 Curves, Chain Rule, Arc Length

## 3. Week: Linearization and Gradient

Lect 9-10 7/6 Partial Derivatives, Review

Lect 11-12 First hourly. Gradient, Linearization

## 4. Week: Extrema and Double Integrals

Lect 13-14 7/13 Tangents, Extrema

Lect 15-16 7/15 Lagrange . Double integrals

## 5. Week: Multiple Integrals and Line Integrals

Lect 17-18 7/20 double and triple integrals

Lect 19-20 7/22 Second hourly. Line integrals

## 6. Week: Vector Fields and Integral Theorems

Lect 21-22 7/27 Curl, Greens theorem, Flux

Lect 23-24 7/29 Stokes and Divergence theorem

## A solid single variable calculus background

### DIFFERENTIATION RULES

- General notes.** In the following, assume  $f$  and  $g$  are differentiable. Each rule should be viewed as saying that the function to be differentiated is differentiable on its domain and that the derivative is as given. For each, there is also a functional form, e.g.,  $(cf)' = cf'$ , and a Leibniz form, e.g.,  $\frac{d}{dx}(u/v) = \frac{u'v - uv'}{v^2}$ .
- Sum.**  $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
- Scalar multiple.**  $\frac{d}{dx}(cf(x)) = cf'(x)$
- Product.**  $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
- Quotient.**  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
- The Chain Rule (for compositions).**  $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ . This says that a small change in input to the composition is scaled by  $g'(x)$ , then by  $f'(g(x))$ . Leibniz notation, if  $x = g(t)$  and  $y = f(g)$ , and we thereby view  $z$  a function of  $t$ , then  $\frac{dz}{dt} = \frac{dy}{dx} \frac{dx}{dt}$ , being evaluated on  $y = g(x)$ . In  $D$  notation,  $D_t(g) = (D_t g) \circ g(D_t)$ .
- Inverse functions.** If  $f$  is the inverse of a function  $g$  (and  $f$  is continuous and nonzero), then  $\frac{d}{dx}f(g(x)) = \frac{1}{g'(f(x))}$ . To get a specific formula directly, start with  $y = f(x)$ ; rewrite it as  $x = g(y)$ ; differentiate with respect to  $y$  to get  $g'(y) = 1/x$ ; write this  $y = g(x)$  and put  $g'(y)$  in terms of  $x$ , using the relation  $y = f(x)$  and  $g'(y) = 1/x$ . E.g.,  $y = \ln x$ ,  $e^y = x$ ,  $e^{f(x)} = x$ ,  $f'(x) = 1/x$ .
- Implicit functions.** The derivative of a function defined implicitly by a relation  $F(x, y) = 0$  may be found by differentiating the relation with respect to  $x$  while treating  $y$  as a function of  $x$  whenever  $y$  appears in the relation, and then solving for  $y'$  in terms of  $x$  and  $y$ . The result is the same as obtained from the formal expression  $y' = -\frac{F_x(x, y)}{F_y(x, y)}$ , where  $x$  is treated as a constant in the numerator,  $x$  as a constant in the denominator.

### DERIVATIVE FORMULAS

- Constants.** For any constant  $C$ ,  $\frac{d}{dx}C = 0$ .
- Reciprocal function.**  $\frac{d}{dx}\frac{1}{f(x)} = -\frac{f'(x)}{f(x)^2}$
- The chain rule gives**  $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$
- Square root.**  $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
- Powers.** For any real value of  $n$ ,  $\frac{d}{dx}x^n = nx^{n-1}$ , valid where  $x \neq 0$  is defined. The chain rule gives  $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$ .
- Exponentials.** An exponential function has derivative proportional to itself, the proportionality factor being the natural logarithm of the base:  $\frac{d}{dx}a^x = a^x \ln a$ ,  $\frac{d}{dx}e^x = e^x$ ,  $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ . The chain rule gives  $\frac{d}{dx}a^{f(x)} = e^{f(x)} \ln a f'(x)$ .
- Logarithms.**  $\frac{d}{dx}\ln|x| = \frac{1}{x}$ ,  $\frac{d}{dx}\log_a|x| = \frac{1}{x \ln a}$ . Same rules hold without absolute value, but the domain is restricted to  $(0, \infty)$ . The chain rule gives  $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$ .
- Hyperbolic functions.**  $\sinh' x = \cosh x$ ,  $\cosh' x = \sinh x$ ,  $\operatorname{arcsinh}' x = \frac{1}{\sqrt{1+x^2}}$ ,  $\operatorname{arccosh}' x = \frac{1}{\sqrt{x^2-1}}$ .
- Trig functions.**  $\sin' x = \cos x$ ,  $\cos' x = -\sin x$ ,  $\sec' x = \sec^2 x$ ,  $\csc' x = -\csc^2 x$ ,  $\operatorname{arcsin}' x = \frac{1}{\sqrt{1-x^2}}$ ,  $\operatorname{arccos}' x = \frac{-1}{\sqrt{1-x^2}}$ .

### ANALYSIS

#### LOCAL FEATURES OF FUNCTIONS

- Neighborhoods.** In the following, "near" a point means in an open interval containing the point. Such an open interval is often called a neighborhood of the point.
- Continuity.** If a function is differentiable at a point, then it is continuous there.
- Critical points.** A point  $c$  is a critical point of  $f$  if  $f$  is defined near  $c$  and either  $f'(c) = 0$  or  $f'(c)$  does not exist.
- Local extremes.** A local minimum point of  $f$  is a point  $c$  with  $f(x) \geq f(c)$  for  $x$  near  $c$ . A local maximum point of  $f$  is a point  $c$  with  $f(x) \leq f(c)$  for  $x$  near  $c$ . If  $c$  is a local extremum point, then it is a critical point. (This follows from definitions.) **Relative extremes** are the same as local extrema.
- First Derivative Test.** Suppose  $c$  is a critical point of  $f$  and  $f$  is continuous at  $c$ . If  $f'$  changes sign from negative to positive as  $x$  increases through  $c$ , then  $c$  is a local minimum point. If  $f'$  changes sign from positive to negative as  $x$  increases through  $c$ , then  $c$  is a local maximum point. If  $f'$  keeps the same sign, then  $c$  is not an extremum point.
- Second Derivative Test.** Suppose  $f$  is differentiable near a critical point  $c$ . If  $f''(c) < 0$ , then  $c$  is a local maximum point. If  $f''(c) > 0$ , then  $c$  is a local minimum point.
- Inflection points.** If the graph of  $f$  has a tangent line (possibly vertical) at  $c$  and  $f'(c)$  changes through  $c$ , then  $c$  is on the graph point  $(c, f(c))$ , is called an inflection point. E.g.,  $x^3$  has a vertical tangent and inflection point at  $(0, 0)$ . An inflection point for  $f$  is an extremum for  $f'$ ; the tangent line is locally steepest at such points. The only possible inflection points are where  $f''(x) = 0$  or  $f''(x)$  does not exist.

#### APPLICATIONS

- Optimization with constraint.** Here is an outline to approach optimization problems involving two variables that are somehow related.
  1. Visualize the problem and name the variables.
  2. Write down the objective function—the one to be optimized—as a function of two variables.
  3. Write down a constraint equation relating the variables in terms of one variable.
  4. Use the constraint to rewrite the objective function in terms of one variable.
  5. Analyze the new function of one variable to find the optimal point(s), and the optimal value. E.g., to maximize the area of a rectangle with perimeter being  $c$ , we pose the problem as maximizing  $A = bc$  subject to the constraint  $2b + 2c = c$ . The constraint gives  $b = c/2 - 1$ , whence  $A = (c/2 - 1)c$ . The maximum occurs at  $c = 2$ , with  $A = 1$  (sq. unit). A verbal result is clearer: it's a square.
- For geometric problems, volume formulas may be needed:** cylinder:  $V = \pi r^2 h$ , cone:  $V = \frac{1}{3}\pi r^2 h$ , sphere:  $V = \frac{4}{3}\pi r^3$ .
- Cables.** A cable  $y(x) = at^2 + bx^2 + c$  has exactly one inflection point:  $(b, k)$  where  $b = -b(3a) + c = k(4)$ . A normal force is  $p(x) = ax + by = m(x - b) + k$  where  $m = b + c$  is the slope at the inflection point. If  $m$  and  $b$  have opposite signs, the horizontal line through the inflection point meets the graph at two points, each a distance  $\frac{c}{2a}$  from the inflection point, and local extrema  $\sqrt{3} = \frac{b}{2a}$  occur at points  $\pm \frac{c}{2a}$  from the inflection point.

#### INTEGRATION

##### INTERPRETATIONS

- Area under a curve.** The integral of a nonnegative function over an interval gives the area under the graph of the function.
- Average value.** The average value of  $f$  over an interval  $[a, b]$  may be defined by  $\frac{1}{b-a} \int_a^b f(x) dx$ . Often a rough estimate of an integral can be made by estimating the average value (by inspection of the graph, say) and multiplying by the length of the interval.
- Accumulated change.** The integral of a rate of change gives the total change in the original quantity over the time interval. E.g., if  $v(t) = v'(t)$  represents velocity, then  $\int_a^b v(t) dt$  is the approximate displacement over the time increment  $t = b - a$ . Adding the displacements for all the time increments gives the approximate change in position over the entire time interval. In the limit of small time increments, one gets the integral  $\int_a^b v(t) dt = s(b) - s(a)$ , which is the total displacement.

#### FUNDAMENTAL THEOREM OF CALCULUS

- Antiderivatives.** An antiderivative of a function  $f$  is a function  $F$  whose derivative is  $f(x) = F'(x)$  for all  $x$  in some domain. Any two antiderivatives of a function on a time interval differ by a constant. (This follows from MVT.) E.g.,  $\arctan x$  and  $-\arctan(1/x)$  are both antiderivatives of  $1/(1+x^2)$  for  $x > 0$ . (They differ by  $\pi/2$ .) An antiderivative is also called an indefinite integral. (The latter term often refers to the entire family of antiderivatives.)
- The Fundamental Theorem.** There are two parts:
  1. **Evaluating integrals.** If  $f$  is continuous on  $[a, b]$ , and  $F$  is any antiderivative of  $f$  on that interval, then  $\int_a^b f(x) dx = F(b) - F(a)$ .
  2. **Constructing antiderivatives.** If  $f$  is continuous on  $[a, b]$ , then the function  $G(x) = \int_a^x f(t) dt$  is an antiderivative of  $f$  on  $[a, b]$  (the one-sided derivative of  $G$  agree with  $f$  at the endpoints).
- Differentiation of integrals.** To differentiate a function such as  $\int_a^x f(t) dt$ , view it as a composition  $G(x) = G(x)$ , with  $G$  as above. The chain rule gives  $G'(x) = f(x)$ ,  $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ .

Note that the domain of  $y = x^p$  is all real numbers while the domain of  $y = a \log_e x$  is  $(0, \infty)$ . In particular,  $\log_e x$  where  $x = 2.718$  is written as  $\ln x$ .

- First class
- Mathematica project due
- Midterm exams
- Independence day
- Final Exam
- Review

Symmetric about the x-axis  
Note: This is not a function of  $x$ .

Symmetric about the origin  
Note: This is not a function of the graph.

Horizontal Asymptote:  $y = 2$

Parametric Functions  
 $y = \frac{x^2 - a^2}{x^2 + a^2}$ ,  $a$  is the parameter  
5. Plot a few more additional points.

**TRANSCENDENTAL FUNCTIONS**

**1. TRIGONOMETRIC FUNCTIONS**  
 $\sin x, \cos x, \tan x, \csc x, \sec x, \operatorname{csc} x$ , where  $\sec x = 1/\cos x$ ,  $\csc x = 1/\sin x$ ,  $\tan x = \sin x/\cos x$ ,  $\cot x = \cos x/\sin x$ .

**2. Key identities:** For any angle  $x$   
 $\sin^2 x + \cos^2 x = 1$ ,  $\sec^2 x = 1 + \tan^2 x$ ,  $\csc^2 x = 1 + \cot^2 x$ ;  
 $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ ,  $\sin 2x = 2 \sin x \cos x$ .

**3. Graphs of trigonometric functions:** In sketching graphs of functions of the form  $y = a \cos(bx + c)$ ,  $y = a \sin(bx + c)$  isolate:  
 $a$  = amplitude  
 $c/b$  = phase angle,  $-c/b$  = phase shift  
 $2\pi/b$  = period.  
 Example: Sketch  $y = 3 \cos(2x + \pi/2)$   
 $y = 3 \cos(2x + \pi/2)$

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