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Chapter 3. Linearization and Gradient

Section 3.1: Partial derivatives and partial differential equations

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

One also uses the short hand notation $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. For iterated derivatives, the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

The notation for partial derivatives $\partial_x f, \partial_y f$ were introduced by Jacobi. Josef Lagrange had used the term "partial differences". Partial derivatives measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

Example: Lets take the function $f(x,y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**: it is an equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to more than one variables.

Clairot's theorem: If f_{xy} and f_{yx} are both continuous, then

$$f_{xy} = f_{yx} \; .$$

A proof is obtained by comparing the two sides for fixed $dx > 0, dy > 0$.

$dx f_x(x,y) \sim f(x+dx,y) - f(x,y)$
$dy dx f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y+dy) - (f(x+dx,y) - f(x,y))$

$dy f_y(x,y) \sim f(x,y+dy) - f(x,y)$.
$dx dy f_{yx}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y) - (f(x,y+dy) - f(x,y))$

If f_{xy}, f_{yx} are both continuous, we can go to the limits $dx \to 0, dy \to 0$ and get Clairot's theorem.

The continuity assumption is necessary: the example

$$f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$$

contradicts Clairaut's theorem:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,

$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**. Examples of partial differential equations are the **wave equation** $f_{xx}(x,y) = f_{yy}(x,y)$ and the **heat equation** $f_x(x,y) = f_{xx}(x,y)$. An other example is the **Laplace equation** $f_{xx} + f_{yy} = 0$ or the **advection equation** $f_t = f_x$.

Paul Dirac once said: "A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed.

We will not learn here to find solutions to partial differential equations. But you should be able to verify that a given function is a solution of the equation:

Here is a problem: verify that $f(x,t) = e^{-rt}\sin(x+ct)$ satisfies the advection partial differential equation $f_t(x,t) = cf_x(x,t) - rf(x,t)$.

Section 3.2: Linear approximation and tangents

The **linear approximation** of a function $f(x)$ at a point x_0 is defined as the linear function

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \; .$$

In two dimensions, the **linear approximation** of a function $f(x,y)$ at (x_0,y_0) is defined as the linear function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \; .$$

The graph of L is tangent to the graph of f. The **linear approximation** of a function $f(x,y,z)$ at (x_0,y_0,z_0) is

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) \; .$$

In three dimensions, the level surface of L is tangent to the level surface of f at (x_0,y_0,z_0).

Using the **gradient**

$$\nabla f = \langle f_x, f_y \rangle$$

or $\nabla f = \langle f_x, f_y, f_z \rangle$ in three dimensions, the linearization can be written as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) \; .$$

Example: lets compute the linear approximation of the function $f(x,y) = \sin(\pi xy^2)$ at the point $(1,1)$. We have $(f_x(x,y), y_f(x,y) = (\pi y^2\cos(\pi xy^2), 2y\pi\cos(\pi xy^2))$ which is at the point

$(1,1)$ equal to $\nabla f(1,1) = \langle \pi\cos(\pi), 2\pi\cos(\pi)\rangle = \langle -\pi, 2\pi\rangle$.

Linearization is important because linear functions are easier to deal with. Using linearization, one can estimate function values near known points.

How do we justify the linearization? If the second variable $y = y_0$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943407 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942478 .$$

Here is an example in three dimensions: find the linear approximation to $f(x, y, z) = xy + yz + zx$ at the point $(1, 1, 1)$. Since $f(1, 1, 1) = 3$, and $\nabla f(x, y, z) = (y + z, x + z, y + x)$, $\nabla f(1, 1, 1) = (2, 2, 2)$. we have $L(x, y, z) = f(1, 1, 1) + (2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$.
An estimation problem: estimate $f(0.01, 24.8, 1.02)$ for $f(x, y, z) = e^x\sqrt{y}z$.

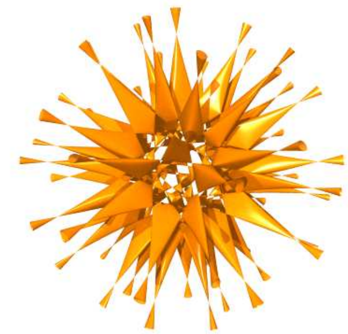
Solution: take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. The gradient is $\nabla f(x, y, z) = (e^x\sqrt{y}z, e^x z/(2\sqrt{y}), e^x\sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b\rangle$ is perpendicular to the level curve $f(x, y) = c$ through (x_0, y_0), the equation for the tangent line is $ax + by = d$ with $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $d = ax_0 + by_0$.

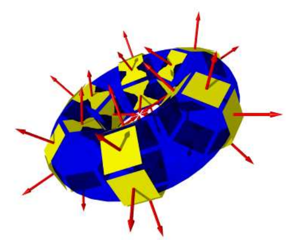
Example: find the tangent to the graph of the function $g(x) = x^2$ at the point $(2, 4)$.
Solution: the level curve $f(x, y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2, 4)$ is obtained by computing the gradient $\langle a, b\rangle = \nabla f(2, 4) = \langle -g'(2), 1\rangle = \langle -4, 1\rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

If $f(x, y, z)$ is a function of three variables, then $\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z))$ is called the **gradient** of f.

If $f(x, y, z)$ is a function of three variables, then $f(x, y, z) = c$ is a level surface. An example is the **Barth surface** $f(x, y) = (3 + 5t)(-1 + x^2 + y^2 + z^2)^2\ (-2 + t + x^2 + y^2 + z^2)^2\ 8(x^2 - t^4y^2)(-t^4x^2 + z^2)\ (y^2 - t^4z^2)\ (x^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4) = 0$, where $t = (\sqrt{5} - 1)/2$ is the golden ratio. Here is a picture of this complicated but beautiful surface:



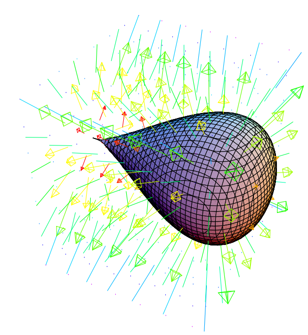
Because $\vec{n} = \nabla f(x_0, y_0, z_0) = \langle a, b, c\rangle$ is perpendicular to the level surface $f(x, y, z) = C$ through (x_0, y_0, z_0), the equation for the tangent plane is $ax + by + cz = d$ with $(a, b, c) = \nabla f(x_0, y_0, z_0)$, $d = ax_0 + by_0 + cz_0$.



Example: The quartic surface

$$f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2, 2, 2)$ of this pair shaped surface? We get $\langle a, b, c\rangle = \langle 20, 4, 4\rangle$ and so the equation of the plane $20x + 4y + 4z = 56$, where we have obtained the constant to the right by plugging in the point $(x, y, z) = (2, 2, 2)$.



Section 3.3: Chain rule and implicit differentiation

If f and g are functions of one variable t, the chain rule tells us that $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$.
It can best be proven by linearizing the functions f and g and verifying the chain rule in the linear case. The **chain rule** is also useful:

For example, to find $\arccos'(x)$, we write $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))} \arccos'(x) = \sqrt{1-x^2} \arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

Define the **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$. If $\vec{r}(t)$ is curve in space and f is a function of three variables, we get a function of one variables $t \mapsto f(\vec{r}(t))$. The **multivariable chain rule** is

$$\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)}$$

When written out it becomes

$$\frac{d}{dt} f(x(t),y(t)) = f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t) \ .$$

Example. A spider moves along a circle $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$ on a table with temperature distribution $f(x,y) = x^2 - y^3$. Find the rate of change of the temperature, the spider experiences:
$\nabla f(x,y) = (2x, -3y^2)$, $\vec{r}'(t) = (-\sin(t), \cos(t))$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

From $f(x,y) = 0$ one can express y as a function of x. From $d/df(x,y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain $y' = -f_x/f_y$. Even so, we do not know $y(x)$, we can compute its derivative!
Implcit differentiation works also in three variables. The equation $f(x,y,z) = c$ defines a surface. Near a point where f_z is not zero, the surface can be described as a graph $z = z(x,y)$. We can compute the derivative z_x without actually knowing the function $z(x,y)$. To do so, we consider y a fixed parameter and compute using the chain rule

$$f_x(x,y,z(x,y))1 + f_z(x,y)z_x(x,y) = 0$$

so that $z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$.

Example: The surface $f(x,y,z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x,y)$ at the point $(x,y,z) = (2,1,1)$.

Solution: $z_x(x,y) = -f_x(2,1,1)/f_z(2,1,1) = -4/(2/9) = -18$.

The chain rule is powerful because it implies other differentation rules like the addition, product and quotient rule in one dimensions: $f(x,y) = x+y, x = u(t), y = v(t), d/dt(x+y) = f_x u' + f_y v' = u' + v'$.
$f(x,y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x,y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

As in one dimensions, the chain rule follows from linearization. If f is a linear function $f(x,y) = ax + by - c$ and if the curve $\vec{r}(t) = \langle x_0 + tu, y_0 + tv\rangle$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0+tu) + b(y_0+tv)) = au + bv$ and this is the dot product of $\nabla f = (a,b)$ with $\vec{r}\,'(t) = (u,v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

Section 3.4: Gradient and directional derivative

The **gradient** of a function $f(x,y0$ is defined as

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$

For functions of three dimensions, we define

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \ .$$

The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. Here is a very important fact:

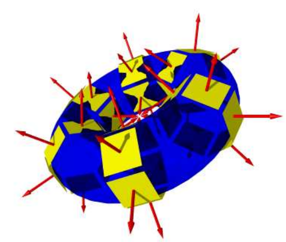
Gradients are orthogonal to level curves and level surfaces.

Proof. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$.

Because $\vec{n} = \nabla f(x_0,y_0) = \langle a,b\rangle$ is perpendicular to the level curve $f(x,y) = c$ through (x_0,y_0), the equation for the tangent line is $ax + by = d$, $a = f_x(x_0,y_0)$, $b = f_y(x_0,y_0)$, $d = ax_0 + by_0$. Ccompactly written, this is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

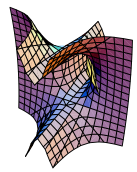
and means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane. It is one of the most important statements in multivariable calculus. since it provides a crucial link between calculus and geometry.



The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines:

Example: Compute the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$.

Solution: $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z\rangle$. And $\nabla f(1,1,1) = \langle 6,3,2\rangle$. The plane is $6x + 3y + 2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x + 3y + 2z = 11$.



Here is an other problem which uses gradients:

Problem: reflect the ray $\vec{r}(t) = \langle 1-t, -t, 1 \rangle$ at the surface

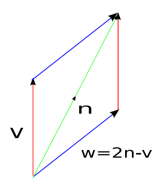
$$x^4 + y^2 + z^6 = 6 .$$

Solution: $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1, -2, 1)$. The velocity vector in that ray is $\vec{v} = \langle -1, -1, 0 \rangle$ The normal vector at this point is $\nabla f(-1, -2, 1) = \langle -4, 4, 6 \rangle = \vec{n}$.

The reflected vector is

$$R(\vec{v} = 2\mathrm{Proj}_{\vec{n}}(\vec{v}) - \vec{v} .$$

We have $\mathrm{Proj}_{\vec{n}}(\vec{v}) = 8/68\langle -4, -4, 6 \rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4, -4, 6 \rangle - \langle -1, -1, 0 \rangle$.



If f is a function of several variables and $\vec{v}$ is a unit vector then $\nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$. One writes $\nabla_{\vec{v}} f$ or $D_{\vec{v}} f$. The name directional derivative is related to the fact that every unit vector gives a direction.

$$\boxed{D_v f(x, y, z) = \nabla f(x, y, z) \cdot v}$$

If $\vec{v}$ is a unit vector, then the chain rule tells us $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

The directional derivative tells us how the function changes when we move in a given direction. Assume for example that $T(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}} T$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $h(x, y)$, then $D_{\vec{v}} h(x, y)$ gives us the slope

we drive on.

In any case, if $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

Because for $\vec{v} = (1, 0, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$, the directional derivative is a generalization of the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.
The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|}$ because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}||\cos(\phi)| \leq |\nabla f||\vec{v}|$.
The direction $\vec{v} = \nabla f / |\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

If $\vec{v} = \nabla f / |\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

Example: You are on a trip in a air-ship over Cambridge at $(1, 2)$ and you want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?

Solution: Parameterize the direction by $\vec{v} = \langle \cos(\phi), \sin(\phi) \rangle$. The pressure gradient is $\nabla p(x, y) = (2x, 4y)$. The directional derivative in the ϕ-direction is $\nabla p(x, y) \cdot v = 2\cos(\phi) + 4\sin(\phi)$. This is maximal for $-2\sin(\phi) + 4\cos(\phi) = 0$ which means $\tan(\phi) = 1/2$.

The directional derivative has the same properties than any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f + g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + fD_v(g)$.

Example. The Matterhorn is a 4'478 meter high mountain in Switzerland. It is quite easy to climb with a guide because there are ropes and ladders at difficult places. Evenso there are quite many climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by the function $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the south-east direction $v = \langle 1, -1 \rangle / \sqrt{2}$. Calculate the rate of change in that direction. We have $\nabla f(x, y) = \langle -2x, -2y \rangle$, so that $\langle 20, -20 \rangle \cdot \langle 1, -1 \rangle / \sqrt{2} = 40/\sqrt{2}$. This is a place, with a ladder, where you climb $40/\sqrt{2}$ meters up when advancing 1m forward.

The rate of change in all directions is zero if and only if $\nabla f(x, y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f / |\nabla f|$ and get $D_{\nabla f} f = |\nabla f|$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = (0, 0)$ are called **critical points** and help to understand the function f.

Problem. Assume we know $D_v f(1, 1) = 3/\sqrt{5}$ and $D_w f(1, 1) = 5/\sqrt{5}$, where $v = \langle 1, 2 \rangle / \sqrt{5}$ and $w = \langle 2, 1 \rangle / \sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f.

Solution: Let $\nabla f(1, 1) = \langle a, b \rangle$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.