Math 21b Midterm 1 Review

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This material is meant to suggest topics to review. There are certainly other topics that may be covered on the exam, but these are a good place to start studying.

1 Linear Equations

1.1 Introduction to Linear Systems

- Geometric interpretation - the solution to a linear system in n variables can be interpreted geometrically by thinking about n-dimensional space and the possible ways for different objects to intersect. Some example pictures for 3-dimensions are on page 4 of the text.
- A system of equations can have 0, 1, or infinitely many solutions.

1.2 Matrices and Gauss-Jordan Elimination

- Matrix - an array of numbers, consisting of rows and columns. A number or variable occupying a particular position in a matrix is called an *entry* or a *component*. The entry in the i^{th} row and j^{th} column of a matrix is called the ij^{th} entry.
- Vector - a matrix with only one column. The columns of a matrix are sometimes called the *column vectors*, and a matrix with only one row is called a *row column*. Note, if neither *row* nor *column* is specified, the vector is understood to be a column vector.
- Coefficient matrix - the matrix whose entries are the coefficients of a system of linear equations, where each row represents a different equation and each column represents the coefficients of a different variable.
- Augmented matrix - adding a column vector to the right side of a coefficient matrix where that column vector represents the vector $\vec{b}$ in $A\vec{x} = \vec{b}$.
- Gauss-Jordan Elimination - a method for reducing any matrix to a form readily displaying much information about the transformation represented in the original matrix. For the actual algorithm, see page 17.
- Reduced row-echelon form (or *rref*) - a matrix where each row's first nonzero entry is a 1 (called the *leading 1*), and where all the entries in a column containing a leading 1, apart from this leading 1, are 0's. Also, if a row contains a leading 1, then each row above it has a leading 1 that is further to the left. This form is the end result of Gauss-Jordan Elimination.

- Leading 1 (as opposed to leading variable) - when dealing with a coefficient matrix, all the leading 1's are leading variables. However, in an augmented matrix, a leading 1 in the last column does not represent any variable, but indicates that the system is inconsistent. Suppose this is not the case. Then if all the variables are leading 1's, there is a unique solution to the system, but if any variables are nonleading then there are infinitely many solutions.

1.3 On the Solutions of Linear Systems

- Rank - the rank of a matrix A is the number of leading 1's in $rref(A)$. This is not necessarily the number of leading variables. See Example 3 on page 25.
- A linear system with fewer equations than unknowns has either no solutions or infinitely many solutions. A linear system of n equations with n unknowns has a unique solution if and only if the rank of its coefficient matrix A is n.
- $\mathbb{R}^n$ - the space defined by ordered "n-tuples" of real numbers, e.g. $(x_1, \ldots, x_n)$ where the x_i are real.
- Linear combination - something is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_n$ if it can be written as $k_1\vec{v}_1 + \ldots + k_n\vec{v}_n$ where $k_1, \ldots, k_n$ are scalars.
- Matrix arithmetic - see pages $29-32$.

2 Linear Transformations

2.1 Introduction to Linear Transformations and Their Inverses

- Linear transformation - A function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ where there is an m x n matrix matrix A such that $T(x) = A(x)$. In other words, it is a function whose operation on x can be written as some matrix A times the n-tuple x. See page 46.
- Inverse linear transformation (where it exists) - a transformation that "undoes" whatever the transformation T did to a vector; a way of getting back to the original vector. A matrix A (and hence a transformation T) has an inverse if and only if the transformation matrix A is an n x n matrix (it's square) and $rank(A) = n$.
- Identity matrix I_n - the n x n matrix consisting of all 0's except along the main diagonal which is all 1's.
- The standard (unit) vector $\vec{e}_i$ - the column vector of all 0's except in the i^{th} coordinate which is a 1. These are the vectors that represent the direction of each coordinate axis.

2.2 Linear Transformations in Geometry

- Linear transformation (take 2) - a transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if and only if

  1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all $\vec{v}, \vec{w}$ in $\mathbb{R}^n$.
  2. $T(k\vec{v}) = kT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars k.

- Characterization of linear transformations - determine exactly what sort of geometric interpretation to associate with a given transformation. The possibilities are dilation, rotation, rotation and dilation, reflection, shear, and orthogonal projection.

2.3 The Inverse of a Linear Transformation

- Invertible functions - a function T from X to Y is invertible if the equation $T(x) = y$ has a unique solution x in X for each y in Y.

- Invertible matrices - a matrix A is invertible if the linear transformation $\vec{y} = A\vec{x}$ is invertible. The matrix of the inverse transformation is written A^{-1}. If the transformation $\vec{y} = A\vec{x}$ is invertible, its inverse is $\vec{x} = A^{-1}\vec{y}$. You can decide when a linear transformation is invertible by determining if each output of the transformation came from a unique input (i.e. rank = n). If A is noninvertible, then the system $A\vec{x} = \vec{b}$ has infinitely many solutions or no solutions. If $\vec{b} = \vec{0}$, then $\vec{x} = \vec{0}$ is a solution. In this case, A is invertible if this is the only solution and noninvertible if there are infinitely many other solutions.

- How to find an inverse - augment the matrix A with the identity matrix of the same size and row reduce A, making sure to apply the same operations to the identity matrix as to A. When A is reduced to the identity (which must be possible in order for A to have an inverse; this is equivalent to saying $rank(A) = n$), the matrix on the right hand side, where the identity was originally, is A^{-1}. See page 70 for the explicit formula for A^{-1} when $n = 2$.

2.4 Matrix Products

- Matrix multiplication - first be aware that not all matrices can be multiplied. They need to be of compatible sizes, namely an m x n matrix can only be multiplied on the right by an n x p matrix (or on the left by an r x m matrix) and the outcome will be an m x p matrix (or an r x n matrix). To determine the entry in the i^{th} row and j^{th} column of the product BA, consider the i^{th} row of B and the j^{th} column of A and add the products of their corresponding terms (multiply the first terms in each, add this to the product of the second terms in each, add this to the product of the third terms in each, etc.). See page 78.

- Properties of matrix multiplication:

  1. For an invertible n x n matrix A, $AA^{-1} = A^{-1}A = I_n$.
  2. For an m x n matrix A, $AI_n = I_mA = A$.
  3. Associativity: $(AB)C = A(BC)$.
  4. If A and B are invertible n x n matrices, then BA is invertible, and $(BA)^{-1} = A^{-1}B^{-1}$.
  5. Distributive property: If A and B are m x n matrices and C and D are n x p matrices, then $A(C + D) = AC + AD$ and $(A + B)C = AC + BC$.

  Note. Notice: matrix multiplication is *NOT* always commutative (i.e. it is not always true that $AB = BA$.).

3 Subspaces of $\mathbb{R}^n$ and Their Dimensions

3.1 Image and Kernel of a Linear Transformation

- Image - the image of a function is the set of all values the function takes in the codomain.
- Span - the span of a set of vectors is the set of all linear combinations of those vectors.
- Image of a linear transformation - the span of the columns of A. This is occasionally called the column space of A. It is all the possible outputs of the transformation, and is a subset of the range. The image of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ has the following properties:

    1. The zero vector $\vec{0}$ in $\mathbb{R}^m$ is in $\text{im}(T)$.
    2. The image is closed under addition. I.e., if $\vec{v}_1$ and $\vec{v}_2$ are both in $\text{im}(T)$, then so is $\vec{v}_1 + \vec{v}_2$.
    3. The image is closed under scalar multiplication. I.e., if a vector $\vec{v}$ is in $\text{im}(T)$ and k is an arbitrary scalar, then $k\vec{v}$ is in the image as well.

- Kernel - the kernel of a linear transformation $T(\vec{x}) = A\vec{x}$ is the set of all vectors that T maps to $\vec{0}$. This can also be called the nullspace. Notice that the kernel is a subset of the *domain*. The kernel of a linear transformation T has the following properties:

    1. The zero vector $\vec{0}$ is in $\ker(T)$.
    2. The kernel is closed under addition.
    3. The kernel is closed under scalar multiplication.

- The invertible case - see Fact 3.1.7 on page 105.

3.2 Subspaces of $\mathbb{R}^n$ Bases and Linear Independence

- Subspaces of $\mathbb{R}^n$ - a subset W of $\mathbb{R}^n$ is called a subspace if it has the following properties:

    1. W contains the zero vector in $\mathbb{R}^n$.
    2. W is closed under addition: if $\vec{w}_1$ and $\vec{w}_2$ are both in W, then so is $\vec{w}_1 + \vec{w}_2$.
    3. W is closed under scalar multiplication: if $\vec{w}$ is in W and k is an arbitrary scalar, then $k\vec{w}$ is in W.

- Linear independence - a set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$ is linearly independent if none of the vectors can be written as a linear combination of the others. Otherwise the vectors are *linearly dependent.* There cannot be more linearly independent vectors than there are spanning vectors. Equivalently, if the space is spanned by q vectors, than any set of more than q vectors is linearly dependent.
- Basis - the vectors $\vec{v}_1, \ldots, \vec{v}_m$ as defined above form a basis for V if they span V and are linearly independent.

- Linear relations - say $\vec{v}_1, \ldots, \vec{v}_m$ are vectors in $\mathbb{R}^n$. An equation of the form

$$c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = 0$$

is a linear relation among the vectors $\vec{v}_i$. There is always the trivial relation, with $c_1 = \ldots = c_m = 0$. Nontrivial relations may or may not exist among the vectors $\vec{v}_1, \ldots, \vec{v}_m$. The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ are linearly dependent if and only if there are nontrivial relations among them. See Fact 3.2.6 on page 117.

- Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$. The vectors $\vec{v}_i$ are a basis of V if and only if every vector $\vec{v}$ in V can be expressed *uniquely* as a linear combination of vectors $\vec{v}_i$.

3.3 The Dimension of a Subspace of $\mathbb{R}^n$

- Dimension - the dimension of a subspace V of $\mathbb{R}^n$ is the number of vectors in a basis of V. If $\dim(V) = d$, then:

  1. It is impossible to find more than d linearly independent vectors in V.
  2. No less than d vectors will span V.
  3. If d vectors in V are linearly independent than they form a basis of V.
  4. If d vectors span V then they form a basis of V.

- If A is an m x n matrix A, then $\dim(\ker A) = n - \text{rank}(A)$.

- Pivot column - a column of A is called a pivot column if the corresponding column of $rref(A)$ contains a leading 1. The pivot columns of a matrix A form a basis of $\text{im}(A)$.

- For any matrix A, $\text{rank}(A) = \dim(\text{im}(A))$.

- Rank-Nullity Theorem: If $A$is an m x n matrix, then $\dim(\ker A) + \dim(\text{im}(A)) = n$.

- READ AND KNOW Summary 3.3.11 on page 130.

3.4 Coordinates

- Review Definition 3.4.1.

4 Linear Spaces

4.1 Introduction to Linear Spaces

- Linear space - a linear space V is a set with rules for addition and scalar multiplication satisfying the following rules:

  1. $(f + g) + h = f + (g + h)$.
  2. $f + g = g + f$.

3. There is a *neutral element* n in V such that $f + n = f$ for all f in V. This n is unique and denoted 0.
4. For each f in V there is a g in V such that $f + g = 0$. This g is unique and denoted by $(-f)$.
5. $k(f + g) = kf + kg$.
6. $(c + k)f = cf + kf$.
7. $c(kf) = (ck)f$.
8. $1f = f$.

- Define span, linear independence, basis, and coordinates for general subspaces on page 154.
- Linear differential equations - the solutions of the DE

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \ldots + a_1 f'(x) + a_0 f(x) = 0$$

form an n-dimensional subspace of C^∞. A DE of this form is called an n^{th}-order linear differential equation.

- Finite-dimensional linear spaces - a linear space V is called finite-dimensional if it has a finite basis $f_1, \ldots, f_n$, so that we can define its dimension $\dim(V) = n$.

4.2 Linear Transformations and Isomorphisms

- Define linear transformations, image, kernel, rank, and nullity for general linear spaces on page 159.
- Isomorphisms and isomorphic spaces - an invertible linear transformation is called an isomorphism. We say that two linear spaces are isomorphic if there is an isomorphism between them. See properties of isomorphisms on page 162.

5 Orthogonality and Least Squares

5.1 Orthonormal Bases and Orthogonal Projections

- Orthogonal - two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.
- Length or magnitude or norm - for a vector $\vec{v}$ in $\mathbb{R}^n$, this is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$. A vector with length 1 is called a *unit vector*.
- Orthonormal vectors - the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ are called orthonormal if they are all unit vectors and orthogonal to one another. See page 178 for a neat way of saying this. Orthonormal vectors are linearly independent. If you have n orthonormal vectors in $\mathbb{R}^n$, you have a basis for $\mathbb{R}^n$.
- Orthogonal complement - V is a subspace of $\mathbb{R}^n$ with basis $\vec{v}_1, \ldots, \vec{v}_m$. $V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v}_i \cdot \vec{x} = 0, \text{ for } i = 1, \ldots, m\}$. $V^\perp$ is also a subspace of $\mathbb{R}^n$.

- Orthogonal projection - V a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{v}_1, \ldots, \vec{v}_m$. For any $\vec{x}$ in $\mathbb{R}^n$, there is a unique vector $\vec{w}$ in V such that $\vec{x} - \vec{w}$ is in $V^\perp$. This $\vec{w}$ is the *orthogonal projection* of $\vec{x}$ onto V, denoted $\text{proj}_V \vec{x}$, and

$$\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \ldots + (\vec{v}_m \cdot \vec{x})\vec{v}_m.$$

  Also, if $\vec{v}_1, \ldots, \vec{v}_n$ is an orthonormal basis for $\mathbb{R}^n$, then for all $\vec{x}$ in $\mathbb{R}^n$,

$$\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \ldots + (\vec{v}_m \cdot \vec{x})\vec{v}_m.$$

- Pythagorean theorem - see facts about this on page 184.
- Cauchy-Schwarz inequality - if $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, then $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$. This is an equality if and only if $\vec{x}$ and $\vec{y}$ are parallel.
- Angle - the angle α between two nonzero vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ is

$$\alpha = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}.$$